



**MATHEMATICS**

**PHYSICS**

**COMPUTING**

**AWARENESS**

**RED ROSE**

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**COMPUTING**

**AWARENESS**

**3 in 1**

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**Preface**

The main reason for writing this book is to have and give an idea of a neat and deep consideration on three strongly related topics of math, physics and computing. The audience of this book are people with a background in these topics however a lot of endeavors have been done to bring everything down. Thus a keen general audience can also use this text.

This is a book composed of three separate books, and every book consists of four parts of; thinking, story, elements and riddles. However in the third book this separation are mixed together due to the essence of computing which is in fact a combination of mathematics and physics. Note is that, this book consider the most core idea, subjects and persons. Thus it does not go to the deep details.

Red Rose March 2022

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# **BOOK ONE**

## **Mathematics**

“An equation means nothing to me unless it expresses a  
thought of God.”

**Srinivasa Ramanujan (1887–1920)**

## **Math thinking**

Mathematics deals with ideas which comes from pure thought. Mathematics is the language of science, engineering and our universe. It helps us to have models for prediction of natural phenomena. Beside philosophy, mathematics is the basic block of our understanding of universe. In fact they are king and queen of science and every truly progressed society.

Mathematical ideas have been alive forever after about 5,000 years. Like its ancient time mathematics is still used to analyze land management issues. Ideas such as Pythagorean Theorem that were developed millennia ago are still valid today. Mathematics has developed over time in amazing and various ways.

Today mathematics is used for DNA research, cryptography, internet and websites, image processing and compression, and many other exciting applications.

Thinking, interpretation and talking are our main difference with other animals. As human beings from pre-historic until now we, have been dealing with sources of knowledge, error and ignorance.

"As we learn from our mistakes, our knowledge grows" therefore error and correction are two main part of our way to pure knowledge. Mathematics is not an exception.



Systematic Exploring, questioning, working, visualizing, conjecturing, explaining, generalizing, justifying and proving... are all at

the heart of mathematical thinking. The picture above contains names of great thinkers, mathematicians of 20 century. Their evolutionary works have changed and eased our life. From nineteenth century, good analytic thinking skills are essential for anyone who wants self-growth and advancement in his/her career in a modern or developing society. People who have good analytic thinking skills are able to acquire new specific skills when needed.

Here we seek for nature of mathematical concepts, in a mathematician mind with a philosophical minded observer to present mathematical ideas. Generally, mathematical thinking defined as "using mathematical techniques, concepts, and methods, directly or indirectly, in the problem-solving process". However this is a part of our daily thought i.e. in parts/steps or as a whole. Mathematicians say, logical and global /overall. But which is more helpful in mathematics?

The human mind tends to make errors: errors of fact, judgment, and interpretation. In the step-by-step method we might not notice these errors as a logical consequence. However in overall

framework, if an error leads to a conclusion that does not fit into the big picture of our problem, the conflict will alert us to the possibility of a mistake. In fact a combination of logical and overall understanding is our best chance of detecting mistakes, in all areas of human knowledge.

In another point of view but with the same ending we can talk about inductive and deductive reasoning. **Inductive** reasoning is the process of arriving at a general conclusion based on limited set of observations or specific example.

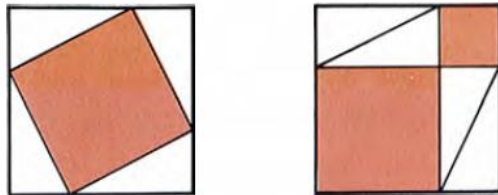
In the path of a ball on a billiard table, an emergency car in traffic, forecasting the weather or matches and stock markets, we collect related evidence, information and data to noticed patterns, and drawn conclusions from these patterns. This method of *inductive* reasoning is used in science mathematics and other fields.

However, the point is that the conclusion if mathematical inductive reasoning are not certain. This is due to the fact that, it is suitable for developing ideas not prove their correctness. "The possibility always exists that additional

evidence will reveal that the conclusions are wrong".

Therefore, conclusions of inductive reasoning are not reliable since they are derived from *suggesting* processes. For certain reliable conclusions mathematicians use a method of deductive reasoning.

**Deductive** reasoning is the process of proving a specific conclusion from one or more general statements. A conclusion that is proved to be true by deductive reasoning is called a **theorem**. In fact, "deductive reasoning is a method of drawing conclusions from facts that we accept as true by using logic". This is the method used by Euclid in his book the Elements around 300 B.C. thus following him this is one of the fundamental reasoning tools in mathematics. Maybe the most famous example is proving Pythagorean Theorem geometrically as below:



The key components of a mathematical theory are axioms, definitions, theorems and proofs. A good understanding of axioms, definitions and theorems equipped one to have a proof without so much difficulties. Mathematicians defined them as below:

An *axiom* is a true statement based on a developed theory.

A *definition* is a precise statement of the meaning of a mathematical term.

A *theorem* is a statement about a relationship among concepts.

A mathematical theorem is composed of *premises* and *conclusions* i.e. suppose/if....be existed or true then...as the conclusion.

A **proof** is a convincing argument or statement that a **theory** is true. Thus a proof must "fit into mathematical theory".

To earn mathematical thinking we need problem-solving skills in different perspectives. You try to solve a problem by finding a relationship between concepts, and thinking begins at this stage. Mathematical thinking is a processes. It is

not about any particular branch of mathematics but there are productive things you can do. Just dare and do it! Since mathematical thinking is a process, it is better to do by examples. Let's start with some IQ tests with taste of math.

**1** The white dot moves two places anti-clockwise at each stage and the black dot moves one place clockwise at each stage. After how many stages will they be together in the same corner?



**2** 72496 is to 1315, and 62134 is to 97, and 85316 is to 167 therefore 28439 is to?

$$1412: 2 + 8 + 4 = 14; 3 + 9 = 12$$

**3** Put the following words into alphabetical order: arthropod, artificer, arteriole, artichoke, arthritis, articular, artillery, arthritic.

Arteriole, arthritic, arthritis, arthropod, artichoke, articular, artificer, artillery.

**4** What numbers should replace the question marks?

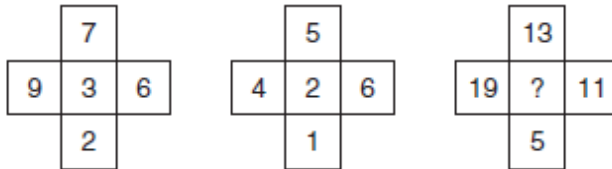
2	6	3	7	?
6	3	6	3	?
3	6	3	6	?
5	2	6	3	?

4,6,3,7.

**5** What numbers should replace the question marks? 100, 95, ?, 79, 68, ?, 40, 23.

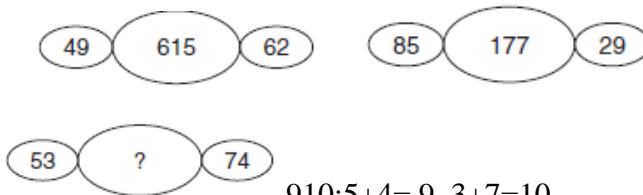
88 and 55: deduct 5, 7, 9, 11, 13, 15, 17

**6** What number should replace the question mark?



$$6: (19 + 11) \div 5 = 6; (13 + 5) \div 3 = 6$$

**7** What number should replace the question mark?



$$910:5+4=9, 3+7=10$$

**8** What number should replace the question mark?

19	9	17
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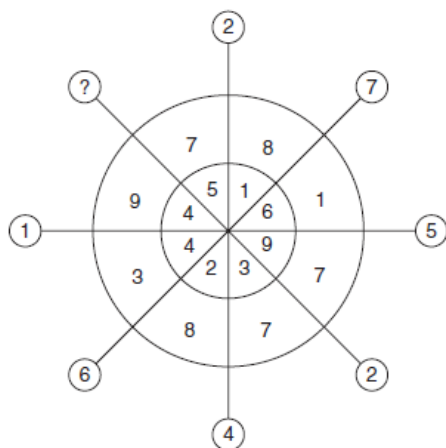
23	12	25
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13	?	31
----	---	----

11:  $13 + 31 = 44$  and  $44 \div 4 = 11$

**9** What number should replace the question mark?

(Each number in the outer circle is the difference of the two numbers immediately next to it anti-clockwise in the inner circles)



9-4=5, 7-5=2, 8-1=7, and so on.

**10** If A=3, B=4, C=6 and D=7, calculate the following: 2

$$\frac{(C \times D) - (B \times C)}{(A + C)}$$

**11** If my taxi journey takes 23 minutes and my train journey takes 49 minutes longer, what is my total travelling time in hours and minutes?

95 minutes, or 1 hour 35 minutes.

**12** In a survey on the High Street on a Saturday afternoon,  $\frac{5}{16}$  of women questioned had bought just cosmetics,  $\frac{5}{8}$  had bought just clothing, while 115 women had just browsed and bought nothing. How many women had just bought cosmetics and how many had just bought clothing?

$\frac{5}{16} + \frac{5}{8} (\frac{10}{16}) = \frac{15}{16}$ :  $\frac{1}{16}$  therefore, = 115 and  $\frac{5}{16}(5 \times 115)$  or 575 had just bought cosmetics;  $\frac{10}{16}(10 \times 115)$  or 1150 had just bought clothing.

By these samples clearly we see that mathematical thinking is a logical process. However, intuition and guessing is possible, but understanding and solving a problem is a logical

process. The logical process yields a reliable and duplicable truth (result). Keep in mind that logic is one of the main pillars of mathematics, we go on with some math puzzles.

**1** If five girls pack five boxes of flowers in five minutes, how many girls are required to pack fifty boxes in fifty minutes?

Five girls pack five boxes in five minutes,  
 Five girls pack one box in one minute  
 Five girls pack fifty boxes in fifty minutes.

**2** A town has a population of 20,000 people. 5 % of them are one-legged, and half the others go barefoot. How many sandals are worn in the town?

Here percentage does not matter and we need for the whole population on the average one shoe per person then 20,000.

**3** Without introducing + signs, arrange six "nines" in such a way that they add up to 100.

$$99\frac{9}{9}$$

**4** A fish had a tail as long as its head plus a quarter the length of its body. Its body was three-quarters

of its total length. Its head was 4 inches long.  
What was the length of the fish?

Let  $H$  represent the head,  $B$  the body,  $T$  the tail, and  $L$  the total length of the fish. Looking at the problem we are given the following three facts:

$$T = H + 1B$$

$$B = 3L$$

$$H = 4 \text{ inches}$$

$$\text{It is also true that } L = H + B + T$$

In this equation keep  $L$  and substitute for everything else.

$$L = 4 \text{ inches} + -L + (H + 1B)$$

$$L = 4 \text{ inches} + -L + 4 \text{ inches} + -3L$$

$$L = 8 \text{ inches} + -4L$$

$$5L = 8 \text{ inches}$$

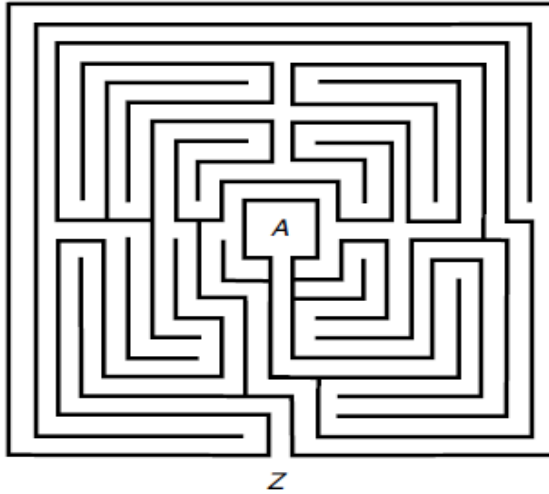
$$L = 8 \times 16 \text{ inches}$$

$$L = 128 \text{ inches}$$

Thus we see that the fish was 128 inches long.

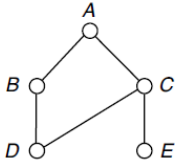
**Note:** In the above sample you can see the logical process to understand a problem then set a logical sketch as an equation then put the corresponding numbers in to your own built equation then solve it and get the final answer. This is the practical logical process to all of our problems.

6 solve the maze below?



7 Represent a social network as a graph with a group of five classmates to work on a class project. A few of them are close friends with one another, while others are not. Anand is friends with **Brittany** and with **Claire**; **Dexter** is also friends with **Brittany** and **Claire**; in addition, **Claire** is a friend of **Ethan**.

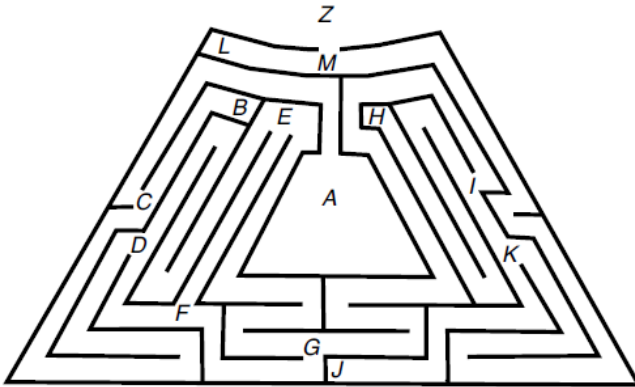
Represent this situation as a graph, with vertices representing the students and an edge indicating friendship between two students. Then determine two ways in which a message



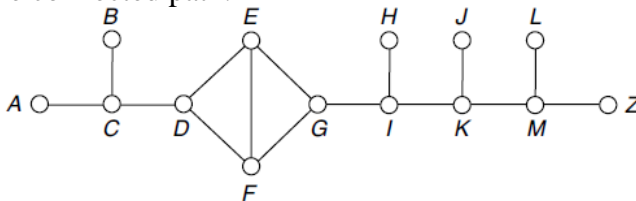
We see that C is adjacent to A, D, and E. The sequence  $A \rightarrow C \rightarrow E$  represents a path of length 2 from Anand to Ethan. A second (longer) path is  $A \rightarrow B \rightarrow D \rightarrow C \rightarrow E$ , of length 4.

**Note** in here we understand the problem then set a sketch as a graph then get the answer.

**8** Solve the Hampton Court Maze Using the Graphical Approach.



First Labels for the Hampton Court maze. Then connect the connected path.



**9** Find all right triangles with sides of integer lengths for which the hypotenuse is one unit longer than one of the legs.

Denote the lengths of the legs by  $a$  and  $b$ ; the length of the hypotenuse is then  $b + 1$ . Pythagoras' theorem gives  $a^2 + b^2 = (b + 1)^2$ , whence  $a^2 = 2b + 1$ , which implies that  $a$  must be odd. Writing  $a = 2k + 1$ , we obtain  $4k^2 + 4k + 1 = 2b + 1$ , whence  $b = 2k(k + 1)$ . Hence there are infinitely many such triangles, and the triples of the lengths of their sides are given by  $(a, b, c) = (2k + 1, 2k(k + 1), 2k^2 + 2k + 1)$ , where  $k$  ranges over the natural numbers. Here are the first few such triples.

$a$	3	5	7	9	11	13	15
$b$	4	12	24	40	60	84	112
$c$	5	13	25	41	61	85	113

**10** The following sequence has become known as the “See and Say Sequence,” so named by noted Princeton mathematician John Horton Conway. Can you determine the next term of the sequence?  
1, 1, 1, 3, 1, 4, 1, 1, 3, 6, 1, 2, 3, 1, 4, 8, 1, 3, 3, 2, 4, 1, 6?

You begin the sequence thus by: 1

Then you read out loud what you see: “one one.” And those are the next two terms of the sequence: 1, 1, 1

And those are the next two terms of the sequence:

1, 1, 1, 3, 1

And those are the next four terms of the sequence:

1, 1, 1, 3, 1, 4, 1, 1, 3

You continue to generate terms of the sequence by seeing and saying. Hence the somewhat eponymous name for this sequence.

The hallmark of a great mathematician is that he/she knows how to seize triumph from the jaws of defeat. And that is just what John Conway did in this situation.

**Conway's Cosmological Theorem:** Every “see and say” sequence eventually splits (“decays”) into a sequence of “atomic elements,” which are finite subsequences that never again interact with their neighbors. There are 92 elements containing the digits 1, 2, and 3 only, which John Conway named after the natural chemical elements. There are also two “transuranic” elements for each digit other than 1, 2, and 3.

The terms eventually grow in length by about 30% per generation. In particular, if  $L_n$  denotes the number of digits of the  $n$ th member of the sequence, then the limit of the ratio  $L_{n+1}/L_n$  exists and is given by

$$\lim_{n \rightarrow \infty} \frac{L_{n+1}}{L_n} = \lambda$$

Note: Lim is the limit and means approaching to a value or point. Here the value of fraction approaches to  $\lambda$  (lambda) while  $n$  goes to infinity.

## Conclusion

If we consider mathematical thinking as a psychological process then we need it to be successful in our various ways, to give space to

others to help them use their own particular talents to build up their mathematical thinking processes, to get aesthetic beauty of the final edifice of formal definition, theorem and proof. The failure of students is due to the epistemological nature of teaching mathematics not because they are not smart enough. In the following books we seek an amusing way to overcome this problem.

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## **Math story**

From 5000 years before in Persian Lands, Elam and Babylon schools have been teaching mathematics. However evidence of using numbers for calculation goes back to the Elamite civilization but one can trace back the counting to our ancestors 11,000 years before, however new research shows that even some animals can count!

From ancient time mathematics developed by our needs to solve our problems with nature or to explore the nature for our curious desires. It started to grow as a child by the hand of ancient developed civilizations such as Persians (Babylonians, Elamites), Greeks, Chinese and Indians. They used their own built geometrical and mathematical rules to explore the sky and nature around, to build magnificent building such as ziggurats pyramids, castles and palaces which still are upright. According to the clay tablets from ancient sites of Iran and Iraq, calculation and writing started from Elamite civilization then come to the Babylon, then after expansion of Persian first empires around 600 BC, reached to its highest level. For example, they create a

symbol for zero! After this time knowledge went to Greece. However Greek's treating with knowledge is very similar to another ancient civilization i.e. Egyptians.

Ramanujan said that only in mathematics could one have a concrete realization of God. His beautiful mind is clear from his intuitive formulas such as his inverse of pi and pi formulas below!

$$\frac{1}{\pi} = \sqrt{8} \sum_{n=0}^{\infty} \frac{(1103 + 26390n)(2n-1)!!(4n-1)!!}{99^{4n+2} 32^n (n!)^3}$$

$$\pi = \frac{5\sqrt{5}}{2\sqrt{3}} \left( \sum_{n=0}^{\infty} \frac{(11n+1) \left(\frac{1}{2}\right)_{Poch(n)} \left(\frac{1}{6}\right)_{Poch(n)} \left(\frac{5}{6}\right)_{Poch(n)} \left(\frac{4}{125}\right)^n}{(n!)^3} \right)^{-1}$$

With this introduction we start our synopsis of history/story of mathematics by its great figures from ancient time to now. However due to the restriction of space we just mention the most prominent ones. Let's start with a big picture of this story then continue with more details.

The history of mathematics can be seen as a series of abstractions started from numbers. We say

"two apples" or "two oranges". Knowing this abstract idea is clear from tallies found on wolf bone and leather knotted bands. They showed that even prehistoric peoples know how to count days, seasons, or years.

We cannot find "mathematics" as we call now until around 3000 BC, when the Babylonians and Egyptians began using arithmetic, algebra and geometry for financial calculations, construction, and astronomy to predict the future events such as Nile flooding events. Many early clay tablets excavated in Elam and Babylon showed that the usage of Pythagorean Theorem and triples goes back to at least 5000 years before. After basic counting or arithmetic and shaping or geometry this is the most ancient and widespread mathematical theorem. It is in Elamite and Babylonian mathematics that elementary arithmetic first appeared in their clay tablets. In contrast of Egyptian inflexible number system they developed a place-value system and used a sexagesimal (base 60) numeral system which is still in use today for measuring angles and time.

Beginning in the 6th century BC Pythagoras spend most of his life in Egypt and Persia. Around

300 BC, Euclid introduced his axiomatic method in *Elements* in thirteen books started from geometry to Algebra and arithmetic. This method, consisting of definition, axiom, theorem, and proof as a four steps logical process. According pre- definitions we suppose something is true as axioms then we grow this axioms as theorems and in the end we prove them. We still using this method today.

Then we reach to the greatest mathematician and physicist of antiquity, Archimedes (287–212 BC). He developed formulas for calculating the surface area and volume of solids of revolution and used the method of exhaustion to calculate the area under the curve, something like the integration methods we use today.

Other notable achievements and developments of Greek mathematics were in, conic sections by Apollonius 300 BC, trigonometry by Hipparchus 200 BC, and Algebra by Diophantus 300 AD.

The numeral system and its operation rules we use today, evolved over the course of the first millennium AD in India and were transmitted to the Western world via Persian mathematician and

scholar Khwarizmi. However people call this number system as Hindi-Arabic or Arabic base 10 system! Newton wrote in Latin but no one call him and his work as Spanish!

Islamic mathematics especially in their Golden Age of 9th and 10th centuries, provided many innovations based on Greek mathematics. They developed algebra, spherical trigonometry and number theory. Many notable mathematicians from this period were Persian and Egyptian origin. Great scholars such as Khwarizmi with his Algebra, Algorithm and perfect square for solving quadratic equations, Khayyam with his binomial theorem, algebra and number theory, Tusi with his contribution to trigonometry and Biruni with his development of geodesics among many others.

With the declining the Islamic power and starting industrial revolution, mathematics began to develop again in Western Europe. Maybe the most outstanding figure is Leibniz and Newton in the 17th century. The prolific Swiss mathematician, Leonhard Euler contribute many notable discoveries in dynamics and number theory in 18th century.

In 19th century German mathematician Carl Friedrich Gauss, made many contributions to algebra, analysis, differential geometry, matrix theory, number theory, and statistics. Kurt Gödel published his incompleteness theorems in the beginning years of 20th century.

Now mathematics has experienced great extensions. The fruitful interactions between mathematics and science (our needs) have raised since the Babylonian and Egyptians time. Mathematical discoveries continue to be made today especially in new mathematical theorems and their proofs.

With this Outline, in the next pages we continue our way based on prominent figures for every era.

The history of mathematics remind us of what we have and teach us how to increase our store of science in general. If numbers are the most basic abstract ideas in mathematics let's start with surrounding lands of Euphrates and Tigris as one of the primeval places of human civilization i.e. Elam and Babylon. They used techniques for predicting astronomical phenomena using mathematical algorithms this is called "mathematical astronomy". Their procedure texts (*epu'su*), about 110 tablets and tabular texts (*tersitu*) about 330 tablets enclose such algorithms and computed tables respectively. Some tablets contain both tables and procedures. With applying procedures for different initial data again and again they compute a synodic table, template table or daily motion table. Their computational systems and concepts contains; connected algorithms and functions and also elementary operations and numbers. They used sexagesimal place-value system which the value of each digit depends on its position within the sequence of base 60. The operations are completely analogous to our own decimal system.

┐	1	┐┐	2	┐┐┐	3	┐┐┐┐	4
┐┐	5	┐┐┐	6	┐┐┐┐	7	┐┐┐┐┐	8
┐┐┐	9	◁	10	◁┐	11	◁┐┐	12
◁┐┐┐	13	◁┐┐	14	◁┐┐┐	15	◁┐┐┐┐	16
◁┐┐┐┐	17	◁┐┐┐	18	◁┐┐┐┐	19	◁◁	20
◁◁◁	30	◁◁◁	40	◁◁◁	50	┐	60

They can be read as ‘1, 40’ means, then, what we would call  $1 \times 60 + 40 = 100$ ; ‘2, 30, 30’ means  $2 \times 60^2 + 30 \times 60 + 30 = 7200 + 1800 + 30 = 9030$ . Thus 60 plays the role which 10 plays in our system.

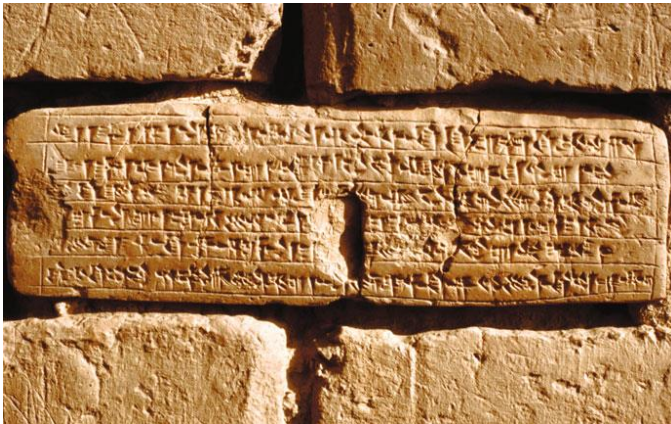
The corpus of Babylonian mathematical astronomy comprises about 440 cuneiform tablets and fragments from the period 450–50 BC whence the region and vast land around it from china's border to Egypt and beyond where ruling by Persian emperors Artaxerxes I, Darius II and Artaxerxes II of Achaemenes. Much light has been thrown on their history by the discovery of the art of reading the cuneiform or wedge-shaped system of writing.



The figure above is a Babylonian tablet called YBC 4663, of the Yale University Babylonian Collection. It contains some algebraic and geometrical problems. In fact opposed to Egyptians, the place value number system provided more elegant facilities for Babylonian mathematics. They used Pythagorean Theorem

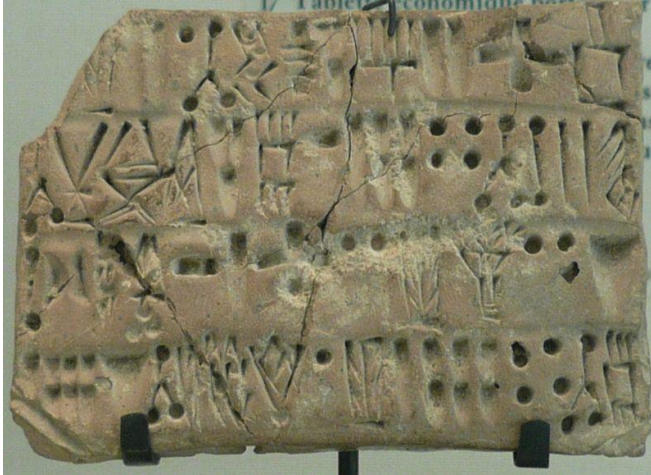
simply because it provides another source of solvable algebraic problems.

However as we said before the writing and number systems were used by Babylonians are strongly affected by their eastern neighboring lands i.e. Elam. Below we have a cuneiform Elamite text. It belongs to 1250 BC written on a brick of Ziggurat of Chogha Zambil-Iran.



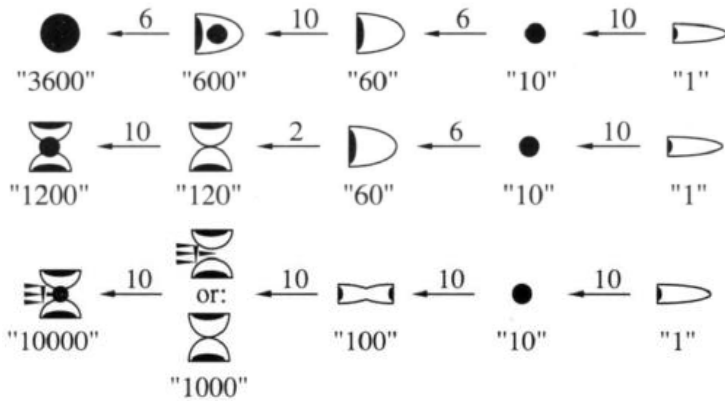
According to "The Proto-Elamite Texts from Tepe Yahya" (Peter Damerow et al, 1989), Harvard University, Even in older periods of Elamite civilization we have *sexagesimal* system and *bisexagesimal* systems as well as *decimal* system of numbers. This is because they have three main class of objects, food and

human/animals. Thus they used this three types of number systems for them respectively. Below we have an Elamite clay tablet.



This is a Proto-Elamite Texts 5000 to 3500 B.C, excavated at Tepe Yahaya. "Studies on an inscription representing an Elamite bill manifest they were the first people in Iran to use a decimal system in their business transactions. Other inscriptions in pictorial language of pre-Elamite show animals, vessels and other objects standing for fractions and decimal numbers,"

In a more clear way we have their number systems of *sexagesimal*, *bisexagesimal*, and *decimal* systems respectively in below;



Egyptians were excellent geometers but not good at Algebra. Maybe this turns back to their holographic number system.



1	2	3	9	10	100	1000	10,000
I	II	III	III III III	∩	9	⌚	⌚



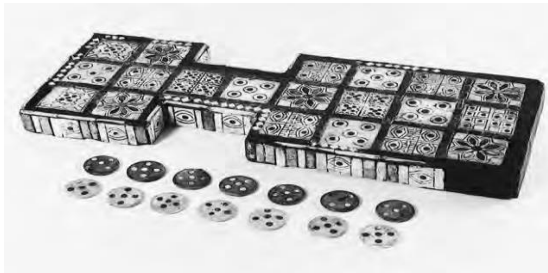
Most of our knowledge of early Egyptian mathematics comes from two papyri, the Rhind Papyrus and the Golenischev or Moscow Papyrus in British Museum and Moscow museum respectively.

The Rhind Papyrus was written about 1650 BC. It is a single scroll nearly 18 feet long and 13 inches high. It contains around 70 problems in numbers and geometry. In below, we see a part of it.



Both Babylonians and Egyptians have some tools for game of chance and probability. In Egypt they use *astragalus*, a bone found in the heels of mammals to produce random patterns. Many of them belong to 5,000 years ago. There are pictures of Egyptians throwing *astragali* while playing board games as well.

A Mesopotamian board game called Game of 20 Squares is excavated in 20<sup>th</sup> century with about 4,500 years old. We know how to play it, because ancient references to the game have also been uncovered. It is played by two people, each of whom relies on a combination of luck and a little strategy to win.



Pythagorean Theorem did not originate with Pythagoras. He contributed its proof, however there are also some earlier proofs. In fact there exist about 70 proofs of the Theorem. It is the

oldest mathematical theorem. It is a source of three mathematical ideas: numbers, geometry, and infinity since it provide a fusion of numbers with geometry! We go on with the Greeks, Muslims, early modern Europeans then come to now.

Our story about **Greek mathematics** happens around the late sixth–fifth century BC, from Athens, *the* cultural center for art, philosophy and mathematics,

The mathematics we find earlier in Egypt or Mesopotamia had weak justification of the method itself. Greek mathematics introduced the quest for general propositions which could be proved. They found the general formula and proved why that formula was right. Until Euclid's *Elements* we have almost no book because no strictly mathematical text has survived from the fifth or fourth century BC. However there is evidence for the use of mathematics in official contexts such as accounting for the expense of building a temple or geometrical town planning. It is obvious that, their nearest neighbor, Egypt have had a strong effect on them. For example as we said before Egyptian number system was

based on their language characters, this is also true about Greek number system. They use their alphabet to say numbers.

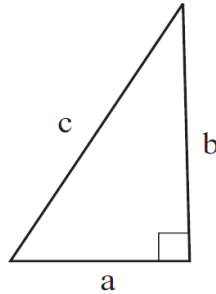
I = 1	$\Delta\Delta\Delta = 30$	$\epsilon = 5$	$\iota\epsilon = 15$
II = 2	H = 100		
$\Gamma = 5$	X = 1,000	$\varsigma = 6$	$\iota\varsigma = 16$
$\Delta = 10$	M = 10,000	$\zeta = 7$	$\iota\zeta = 17$

Actually different forms of mathematics were used for different purposes by different groups of people. It was a public activity. But remember philosophy had a special role in Greek society. Plato's contemporary Diogenes says "the mathematicians should gaze at the sun and the moon, and should neglect music, geometry, astronomy, and the like studies, as useless and unnecessary". Isocrates, says astronomy and geometry could be practiced as gymnastics for the mind, however he did not believe in excessive use of mathematics, since it slow down the melodious mental development of the youth. He did not believe in accuracy more than daily life therefore too much accuracy was not always necessary.

Plato's ideal state was to be ruled by philosophers who are equipped with gymnastics, reading and writing, military training then with mathematics.

He appreciated both the practical and the more philosophical uses of arithmetic. Thus, generally we deal with two kinds of mathematics. Mathematics for daily usage and mathematics for itself, practical and philosophical, applied and pure in our today terms. Let's turn back to mathematics.

According to Walter Burkert, no mathematical discoveries can be soundly attributed to Pythagoras at all. Even his existence is under question. Our evidence to him is referenced back to Herodotus and Heraclitus and not much more than this. By the way it is said that, Pythagoreans contributed two things; first necessity for rigorous proof and the idea that a great body of mathematics (such as geometry) could be derived from a small number of postulates. The second is, he discovered and proved that not all numbers are whole numbers but also there exist fractions of them. Today we called them rational. Pythagoras proved the result that we now call *the Pythagorean Theorem*. It says that the legs  $a$ ,  $b$  and hypotenuse  $c$  of a right triangle are related by the formula:  $a^2 + b^2 = c^2$



This theorem has been proved geometrically and algebraically.

Euclid of Alexandria lived between 325-265 BC. There are not many theorems named after Euclid but he has had a great influence on the realm of mathematics after himself. He wrote a treatise in thirteen Books, known as Euclid's *Elements*. This book has had a substantial influence over the way that we think about mathematics with its axioms, theorems and proofs. The *Elements* develops a large section of elementary geometry by rigorous logic starting from 'undeniable' 5 axioms. According to T. L. Heath, Euclid Elements book I and II.

"Let the following be postulated:

1 - To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles".

Archimedes 287–212 BC Great Greek mathematician who made considerable contributions to geometry, in finding the surface area and the volume of a sphere and the area of a segment of a parabola. His work on hydrostatics and equilibrium was also fundamental in physics then he can be labeled as the first great physicist as well.

His most fascinating work, *The Method*, was rediscovered as recently as 1906. He may or may not have shouted 'Eureka' and run naked through the streets, but he was certainly murdered by a

Roman soldier, an event that marks the end of Greek brilliant era in mathematics. "Stand away, fellow, from my diagram," Archimedes said, whereas the soldier was so enraged that he killed him. One of his invention was that of a sphere constructed so as to imitate the motions of the sun, the moon, and the five planets in the heavens, working with regular circular motion of water. This shows that Archimedes was much occupied with astronomy.

During the **Roman Empire** the atmosphere was dark for foundational scientific research. They just appreciate practical and more military ones. But among a few we can find a woman mathematician Hypatia. She was a Neo-Platonist Greek philosopher who lives in Alexandria during 370–415. One of her quotes is "Reserve your right to think, for even to think wrongly is better than not to think at all". During times of turbulent power struggles between Romans and militant Christians she was martyred by a fanatical mob partly because she did not adhere to strict Christian principles. She edited books on geometry, algebra, and astronomy. Her main focus was on Euclidean geometry and solving

integer equations, and she authored a popular treatise on the conies of Apollonius.

In the ending years of 6 century AD two great empires of Rome and Persia were completely wear out with almost 200 years of war. But from a desert land some people arrived and change the scene for both of them. It takes almost 300 years for them to fix their ruling then translate Greeks works to Arabic and build their scientific tradition. But it is fair to say that, the Idea of Islam was very more attractive than the people carry it. For example Omar the second ruler after the prophet Mohammad, when his army commander, Sad-al-vaghas wrote to him what we must do with books in libraries of Persia and Rome, especially in Jondishapoor and Alexandria he wrote back, Allah's book is sufficient and we do not need to them so burn them all!!

However we do not have almost anything about the science in ancient Persia but fortunately remarkable amount of works in roman lands have preserved from burning and by scholars of Persian and Egyptian origins translated to Arabic and developed to higher levels. During the golden ages they developed them to reach to the hand of

Europe scholars by re-translation of them from Arabic to Latin then to English, German, and Russian etc. Even today this process is continuing.

They developed the decimal place-value number system to include decimal fractions, systematized algebra and began to consider the relationship between algebra and geometry, brought the rules of combinatorics from India and reworked them into an abstract system, studied and made advances on the major Greek geometrical treatises of Euclid, Archimedes and Apollonius, and made significant improvements in plane and spherical trigonometry. A number of medieval thinkers and scientists living under Islamic rule, by no means all of them ‘Moslems’ either nominally or substantially, played a useful role of transmitting Greek, Hindu, Persian and other pre-Islamic fruits of knowledge to Westerners.

Khwarizmi’s *Hisab al-jabr wa al-muqabala* 825 AD is earliest, foundational book on algebra. Much later, and equally important, is the algebra of Khayyam 1070 AD. Both are translated in 19<sup>th</sup> and 20<sup>th</sup> centuries in different European languages. Samaw’al in twelfth century wrote

startlingly innovative algebra text, *al-Bahir fi-l jabr* ('The Shining Treatise on Algebra') in sums of series and with polynomials. Another famous works in algebra, with its sophisticated calculations, is Kashi's *Miftah.al-hisab* ('The Calculation's Key'), fifteenth century. The last one was in Persian but others in Arabic as the scientific language of the period.

Galileo said "Philosophy is written in that vast book which stands forever open before our eyes, I mean the universe; but it cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word". With this we start the period of **scientific revolution** which in it mathematics becomes queen of science. In first two hundred years there was no innovation and the sophisticated Khayyam, or Kashi were not raised. The books were containing, basic calculation with numbers, a little geometry and algebra of the Khwarizmi and

the extraction of square and cube roots were added.

It is around 1500 that the various developments happened. These were in trigonometry from the Islamic, a rapid development in algebra and the general idea of ‘number’, beginnings of a use of the infinitely small or infinitesimal idea associated with Greek tradition, the invention of symbolic algebra, and a primitive calculus.

Renascence started in Italy in 14<sup>th</sup> century. Girolamo Cardano 1501–1576 son of a lawyer in Milan. Being a mathematician by nature, he understood probability theory rather well. Cardano subsequently published numerous other books on mathematics; he wrote on topics as diverse as medicine, astronomy, and philosophy. His masterpiece *Ars Magna* published in 1545. It contains cubic formula for solving cubic equations and the very first calculations with complex numbers. He was a multi-talented individual who made profound contributions mathematics and probability.

Cardano’s method on cubic equations can be extended to fourth, fifth and higher degrees.

Niels Henrik Abel (1802–1829) and Évariste Galois (1812–1832) tried for a formula but discovered their own error. In 1824 Abel proved the solving a quintic equation with a formula involving only arithmetic operations and roots is impossible and there was no elementary formula. The only way however is using **Implicit Function Theorem**. This goes on to the invention of group theory in 19<sup>th</sup> century. However we can find lots of group theory structures in Persian architectures several centuries before proposing the theory, such as below!



René Descartes, 1596–1650 is French philosopher and mathematician. He developed

analytical geometry. In *La Géométrie*, he was concerned to use geometry, to solve algebraic problems used coordinates which called after him Cartesian coordinates.

In 1619 when he was 23, in a letter to Beckman he wrote about new foundations of mathematics to answer any question "that can be put about any kind of quantity whatsoever, whether continuous or discontinuous, each according to its nature". So we can solve Arithmetical problems, by rational numbers, or irrationals, others can be imagined and unsolved. Certain problems can be solved with straight lines and circles alone if the quantity is continuous others can be solved only with curves other than circles, but which can be generated by a single motion.

German mathematician Gottfried Wilhelm Leibniz, (1646–1716), philosopher and scientist in different subjects. Isaac Newton (1642–1727) discovered differential calculus perhaps ten years earlier than Leibniz's, but Leibniz was the first to publish his work independently in 1684. Then, he published a work on integral calculus. This work included the Fundamental Theorem of Calculus. He also helped to the development of symbolic

logic. Unfortunately this one was not followed up until the end of the nineteenth century!

Galileo died on January 8, 1642, Isaac Newton was born on Christmas Day of that year. His father, had died three months earlier. When he was 2 years old his mother remarried and he was placed in the care of his grandmother. At the age of five, he attended school but was considered “inattentive” and “idle”. They put him for managing the farm and estate. But he was no good at as well, so they decided, absent-minded Isaac returned to school to finish his education. Headmaster of the Free Grammar School, John Stokes, led newton to trinity college Cambridge. During the bubonic plague in 1665 university closed. Newton move back to Lincolnshire. In the eighteen months he devoted himself to mechanics and mathematics and concentrated on optics and gravitation. Philosophy of Aristotle and Descartes, mechanics of Copernicus and Galileo’s astronomy, in addition to Kepler’s optics, Euclid’s *Elements*, Algebra of William Oughtred (1574–1660) and François Viète (1540 1603) and, most importantly, Descartes’ *Geometrie* was among the most important works

he studied at Cambridge. François Viète was a French lawyer and also an amateur mathematician and astronomer. He introduced the first systematic algebraic notation in his book *In artem analyticam isagoge*.

The most prolific of famous mathematicians Euler, Leonhard (1707–83). He was born in Switzerland. He made contributions to most areas of mathematics, pure and applied and also the very active topic of the time, calculus. Euler was responsible for notation and symbols that is standard today such as  $\pi$ ,  $e$  and  $i$ , the summation notation  $\Sigma$  and the standard function notation  $f(x)$ . His *introduction to analysis of infinitesimals* was the most important mathematics text of the late eighteenth century. In 1911 the Swiss Academy of Science began to publish his works. Over 72 volumes have been published and more remain! In fact Euler produced nearly a quarter of all of the works in mathematics, physics, mechanics, astronomy, and navigation written in the eighteenth century.

Euler soon became friends with Johann II Bernoulli (1710–1790) and together they attended the lectures of Johann I Bernoulli (1667–

1748), the father of Johann II and brother of Jacob, whose lectures Paulus, his grandfather had attended. Under the influence of both Bernoulli's, the young Euler immersed himself in mathematics. Johann I Bernoulli soon recognized Euler's immense talents and offered him private lectures.

In 1726 Euler completed his *Dissertation on the theory of sound* under the supervision of Bernoulli. This qualified him to apply for the vacant professorship of physics at the University. However the university did not allow and after the establishment of scientific academy in the new Russian capital St. Petersburg In 1724, he and his masters Nicolaus II Bernoulli (1695–1726) and Daniel Bernoulli (1700–1782) attended there as professors. The Petersburg Academy offered Euler its chair in physiology. But he chose to ignore the details of his initial appointment. Instead, he lectured in mathematics, physics, and logic.

Pierre Simon de Laplace (1749–1827) was born in a small town in Normandy. He was a French mathematician, best known for his work on planetary motion, enshrined in his five-volume

*Mécanique céleste*, and for his fundamental contributions to the theory of probability. Laplace recognized that the regularity in the heavens had to have a physical cause and he found it. He achieved what Newton had thought impossible. He demonstrated how the solar system could have arisen from a whirling solar atmosphere with planets whose moons formed as rotating balls of gas that cooled and condensed, with the revolutions necessarily in the same direction in almost the same plane about the sun. It was Laplace who extended Newton's gravitational theory to the study of the whole solar system. He developed the strongly deterministic view that, once the starting conditions of a closed dynamical system such as the universe are known, its future development is then totally determined. He also developed a form of integral transformation with applications in functional analysis, wave equations and also solution to certain types of differential equations. It is called *Laplace transformation*. It is a "description of the original function for  $t > 0$  as the weighted sum of decaying exponential function" with the help of complex variables, line integrals and complex plane. Along with him we have;

French engineer and mathematician, Joseph, Fourier (1768–1830) is best known for his fundamental contributions to the theory of heat conduction which paved the way for significant breakthroughs in pure mathematics and mathematical physics during the remainder of the nineteenth century. His study of trigonometric series which led to so-called *Fourier series* are of immense importance in physics and engineering. The use of Fourier series and Fourier transforms in mathematical analysis for differentiation and integration yield *Fourier analysis*. In 1789, he submitted a paper on a problem in the theory of equations to the Academy of Sciences in Paris. The noted mathematicians Adrien-Marie Legendre and Gaspard Monge recommended the paper for publication in the fall. But on July 14, 1789, a mob stormed the Bastille and the French Revolution began.

In these brilliant years we have German astronomer and greatest mathematician of all time, Carl Friedrich Gauss (1777–1855), made huge contributions in mathematics, physics and astronomy. At the age of 7 when the class began to be unruly, the teacher, J. G. Büttner, assigned

them the task of adding up all of the integers from 1 to 100. As his classmates were struggling he wrote down the answer immediately: 5050. Gauss recognized that the set of integers from 1 to 100 was identical to 50 pairs of integers each adding up to 101:  $(\{1,100\}, \{2,99\}, \dots, \{50,51\})$ . At the age of 18, he invented the method of least squares. At the age of 24, he proved the Fundamental Theorem of Arithmetic and the Fundamental Theorem of Algebra in his book *Disquisitiones arithmeticae* with its deep influence on number theory. Then he developed the theory of curved surfaces using methods now known as differential geometry. His work on complex functions was fundamental but, like his discovery of non-Euclidean geometry, it was not published at the time, since he has a very idealist character. The famous Gaussian distribution in probability and statistics dabbled after him. In astronomy, he calculated the orbits of comets and asteroids from limited observational data.

In nineteenth-century mathematics Bernhard Riemann (1826-66) was German mathematician who was a major figure. In many ways, he was the intellectual successor of Gauss. In geometry,

he started the development of those tools in differential geometry which then called *manifolds* in 20<sup>th</sup> century. Einstein used them to describe his theory of General relativity. He did much significant work in geometry and analysis. His name is preserved in the Riemann integral, the Cauchy-Riemann equations, Riemann surfaces and Riemann geometry as a *non-Euclidean geometry*. "He also made connections between prime number theory and analysis: he formulated the Riemann hypothesis, a conjecture concerning the so-called zeta function, which if proved would give information about the distribution of prime numbers".

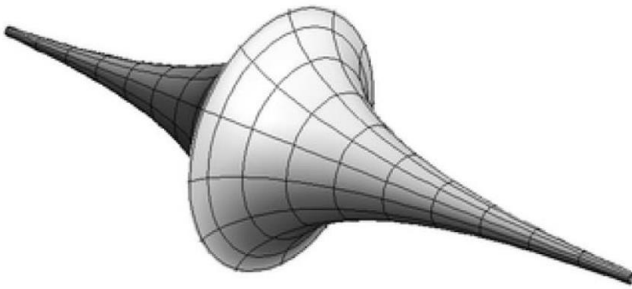
Once Euclid was for a shortcut to geometric knowledge, to which Euclid replied, "there is no royal road to geometry" and also there are no royal road to mathematics as well. But maybe Augustin-Louis Cauchy (1789–1857) is an exception since he was trained lawyer and a commissioner of the Paris police under the royal regime. He was born during French Revolution, on August 21, 1789. He is one of prominent mathematicians of the early nineteenth century in France with 800 papers. His important

contribution is in foundations of rigorous mathematical analysis. He developed sound definitions of *continuity and convergence* by means of definition of limit and also developed the theory of functions of a complex variable.

Last years of 18<sup>th</sup> century and beginning of 19<sup>th</sup> century was a time to end the correctness of 5<sup>th</sup> parallel postulate of Euclid. Gauss, the French Bolyai and Russian Nikolai Ivanovich Lobachevsky, (1792–1856) did individually some works on the subject. But Lobachevsky was more serious and in 1829 independently published his discovery. It was the time for other new geometries. Lobachevsky was born on December 1, 1792 from an Immigrant polish parents in the heart of Russia. In 1814, he appointed as a lecturer at Kazan University. In 1822 he became the full professor there. Lobachevsky lectured on physics and astronomy as well as mathematics, in particular plane and spherical trigonometry was his favorite subject.

Lobachevsky published his essay *Geometrical Researches on the Theory of Parallels*, in 1840." If line  $l_1$  is parallel to line  $l_2$ , then  $l_1$  is *asymptotic* to  $l_2$ , that is  $l_1$  and  $l_2$  come arbitrarily close to each

other. In contrast, in Euclidean geometry, two parallel lines remain equidistant from each other". Italian mathematician Eugenio Beltrami in 1866 verified that Lobachevsky's geometry as the surface of a pseudo-sphere in Euclidean space. A shape as bellow;



French mathematician who had made major contributions to the theory of equations was Galois, Évariste (1811–1832). He lost his life in a stupid, duel in the age of twenty-one. His most important paper, “*Mémoire sur les conditions de résolubilité des équations par radicaux*” was not published until 1846, when his works appeared in Liouville's *Journal de Mathématiques*. His work developed the necessary **group theory** to deal with the question of whether an equation can be solved algebraically. Joseph Liouville, (1809–1882) was editor of a notable French

mathematical journal and proved important results in the fields of number theory, differential equations, differential geometry and complex analysis. In 1844, he proved the existence of infinite class of transcendental numbers.

Not far from the birthplace of the greatest English scientist of all time, Sir Isaac Newton George Boole (1815–1864) was born in Lincoln in the north of England. He then become one of the founding fathers of mathematical logic. In 1854 He published his groundbreaking work in logic *Investigation of the Laws of Thought*. In this book he developed a kind of symbolic argument that has been called Boolean algebras. Boolean algebras have a central role in computing and programming. He says:

"Such general laws as are necessary to constitute a science; for we have seen that it is essentially in recognition of general laws, not of particular facts, that science consists".

His work, along with the works of another British, De Morgan developed the subject as we know today as set theory and logic. Augustus De Morgan, (1806–1871) played a considerable role

in the beginnings of symbolic logic. His name is remembered in De Morgan's laws as; "For all sets  $A$  and  $B$  (subsets of a universal set),  $(A \cup B)' = A' \cap B'$  and  $(A \cap B)' = A' \cup B'$ . He also explained the notion of *mathematical induction*.

In the modern definition of limit and convergence in mathematical analysis, we say; "For every epsilon, there is a delta!" which contributed by Karl Weierstrass, (1815–1897). He was a German mathematician born in Ostenfelde, Westphalia, now a part of Germany, on October 31, 1815. In 1828, Karl entered the Catholic Gymnasium (high school) in the Westphalian town of Paderborn, where he won many mathematics prizes. In one of his exam he made important advances in a branch of mathematics called *elliptical function theory*. He worked more on rigorous mathematical analysis developed out of eighteenth-century calculus. He also made important contributions in expansion of functions in power series. In his youth while he was doing his significant contributions he was just a school teacher however, he was promoted directly to professor of mathematics in Berlin at the age of 40. This is because in 1854, Weierstrass

submitted his paper “*On the Theory of Abelian Functions*” to the most famous mathematical journal, the Journal of Pure and Applied Mathematics. Weierstrass finally received the attention he deserved. The University of Königsberg (now Danzig) offered him an honorary doctorate. He is best known for his “*epsilon* ( $\epsilon$ ) method” in mathematical analysis in 1860 which provides a rigorous way to work with an infinite sequence or series reaching a limit or converging to point or value. His method says; “an infinite sequence as having a limit if for any  $\epsilon$ , you could find an integer  $n$  so that for all integers  $m \geq n$  the  $m$ th term of the sequence was always within the limit not beyond it.

Georg Cantor (1845–1918), was born in St Petersburg but spent most of his life at the University of Halle in Germany. He was one of Weierstrass’s students. He was responsible for the set theory and Infinity ( $\infty$ ) we use today. In 1873, he showed that the set of rational numbers is denumerable. He also showed that the set of real numbers is not finite. Later he fully developed his theory of infinite sets and so-called *transfinite numbers*. Cantor’s work also became the cause of

friction between Weierstrass and one of his oldest and best friends, Leopold Kronecker (1823–1891) on non-constructive existence proofs. Kronecker made considerable contributions to number theory and other fields. Kronecker could accept all sorts of numbers: rational numbers, real numbers, even imaginary numbers but not infinite, the focus of Cantor's work. In Kronecker's mind Cantor's infinite mathematical objects could not exist.

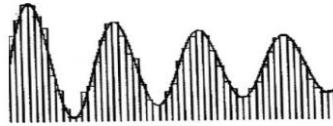
However, in this time, Richard Dedekind, (1831–1916) another German mathematician was developing another foundational development in number theory. In his paper *Continuity and Irrational Numbers* (1872) he contributed a formal construction of the real numbers from the rational which gave the first rigorous foundation for the continuum of real numbers. He also proposed a definition of infinite sets that was taken up by his life-long friend, George Cantor.

Most of the above great figures of modern mathematics were with German origin. Maybe one can call Germany as the new Greece. But others also have effective role on the way. French mathematician, Henri Poincaré, (1854-1912) was

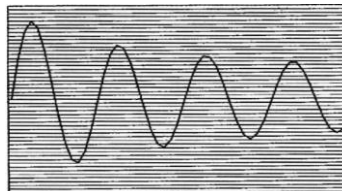
active over the entire field of mathematics since he moved from one area of mathematics to another and made his major contributions. He is one of the main founders of *topology*, and discovered what are called *automorphic functions*. He described qualitative aspects of celestial mechanics and theory of differential equations. Henri Poincare, had high praise for Weierstrass. Another French mathematician, Henri-Léon Lebesgue (1875–1941) born on June 28 1875 in a middle class family, north of Paris. His family encouraged "intellectual pursuit" therefore young Henri had a chance to read widely in their almost large library. However his father died of tuberculosis on those years, the local school which he attended, arranged to finance his higher education because of his brilliant mind. After two years of his graduation from École' in master degree. He found a job in the library as well, so had enough time to work on a new *theory of measure and the integral*, while he pursued his graduate studies at the Sorbonne. He received his doctorate degree from the Sorbonne in 1902. He transfigured the theory of integration. He developed the so-called *Lebesgue integral* on measure theory which is based on set

theory developed by earlier French mathematician. Measure theory and Lebesgue integral are among the outstanding concepts in modern analysis. In fact Lebesgue integral generalizes the Riemann integral being applied just in continuous functions.

In general integration means "add up little changes" or partitions. The integral theory of Cauchy and Riemann was based on the process of partitioning the  $x$ -axis to take a value (changes) of the function  $f(x)$  anywhere on the curve.



While Lebesgue's idea was to partition the  $y$ -axis instead then considered each element of the partition and take the *measure* of that of the portion of the  $x$ -axis that the function being integrated, mapped into that element of the partition.



Therefore Lebesgue had reduced the theory of the integral to the theory of measure since instead of having a two-dimensional object we have a measure (generalized volume) of a set of points in the one-dimensional real line.

Since the discovery of non-Euclidean geometry by Gauss, Lobatchevsky and Bolyai at the beginning of the last century, many new geometrical structures and theories had been developed which have deeply modified and extended the concept of **space**. Mathematicians like Gauss, Riemann, Poincaré, Lie and Hilbert made great contributions to formalizations and geometrization of space in mathematics and physics and also their relations. From the space-time of Minkowski, Einstein, and Weyl to non-Abelian gauge theories, Hilbert and axiomatization of quantum mechanics this process has continued to our times.

The German mathematician David Hilbert (1862–1943) born at Königsberg. He was one of the founding fathers of formalization of pure mathematics in twentieth-century. The formalist school was dominant in the pure mathematics of this century. He became professor at Göttingen in

1895. In his *Grundlagen der Geometrie* (Foundations of Geometry), published in 1899 he made fundamental contributions to formalism. He introduced geometry on a proper axiomatic basis, different from rather more intuitive ‘axiomatization’ of Euclid. He made a major contribution to mathematical analysis. He axiomatized newly born quantum mechanics in physics as well as geometry. At the second International Congress of Mathematics in 1900 in Paris, he opened the new century by posing his famous list of 23 problems. Hilbert became also the chairman of the famous atomic physics seminar at Göttingen that had a great influence on the development of quantum theory.

An eccentric mathematician at Princeton University USA was Kurt Gödel, (1906–1978) became the greatest mathematical logician of all time. He was born on April 28, 1906, in Brno, Moravia, a province of Austria-Hungary, now part of the Czech Republic from a German parents. During 4 years he passed 8 years of school. In 1918 he attended at high school. In the fall of 1924 he went to the University of Vienna where he intended to pursue a career in physics.

On those years the famous Vienna Circle had been established there by a group of scientists and philosophers. They met weekly to set both mathematics and the physical sciences on a firm philosophical foundation. In 1926, as he was seeking a topic for his doctoral dissertation, he attended the Circle's meetings. This turned his attention from physics to logic. His dissertation topic was in "completeness of the predicate calculus". The predicate calculus introduces  $\forall$  (*for all,  $x, y..$* ) as Universal Quantifier and  $\exists$  (*there exist an  $x, y ..$* ) as Existential Quantifiers into logic. His dissertation also introduced *completeness* of the predicate calculus, (*all* true propositions can be proven) and *consistency* of the predicate calculus, (*only* true propositions can be proven). These solved David Hilbert's first two imposed questions on the advancement of science.

In his *Habilitationsschrift*, he showed that the consistency of elementary arithmetic could not be proved from within the system itself. This result followed from his proof that any formal axiomatic system contains undecidable propositions. It undermined the hopes of those

who had been attempting to determine axioms from which all mathematics could be deduced.

However among these western contributions to mathematics we shouldn't forget Indian genius Srinivasan Ramanujan, (1887-1920) born in 22<sup>nd</sup> December 1887. He was a member of a rather poor Brahmin family in the Tanjore District of the Madras Presidency. His father was an accountant to cloth merchants. In his school days students of higher classes go to Ramanujan for the solution of difficult problems. Even in those days he was searching for "highest truth in mathematics. Srinivasan learned totally by himself and unaided. In 1903 when he was in class 6 he gained a copy of "A Synopsis of Elementary Results in Pure Mathematics (1886)" by G. S. CARR, M.A. and worked on it by himself. In December 1903 he passed University of Madras entrance exam, and won a scholarship, which is generally awarded for proficiency in English and Mathematics. After 1907 he had no very definite occupation till 1909, but continued working at Mathematics in his own way and jotting down his results in his famous notebooks. He married 1909 while had not sufficient income. Searching for job

he went to Mr Ramaswami Aiyar the founder of the Indian Mathematical society. Mr Ramaswami Aiyar noticed unusual genius in his notebooks and helped him became an accounting clerk in Madras. During 1910-1912 he publish some articles in *Journal of the Indian Mathematical society*. On the suggestion of Mr Seshu Aiyar his supporter in Madras and others, Ramanujan wrote a letter to Mr G. H. Hardy, Fellow of Trinity College, Cambridge, on the 16th January 1913.

"I had no University education but I have undergone the ordinary school course....I have made a special investigation of divergent series, Orders of Infinity... Being poor... "

Following correspondence with G. H. Hardy, he accepted an invitation to visit Britain in 1914. He studied and collaborated with Hardy on the subject of partitions and other topics, mainly in number theory. On the 28th February 1918, he was the first Indian elected a Fellow of the Royal Society and a Fellow of Trinity college Cambridge. He was considered a genius for his inexplicable ability in, the handling of *series and continued fractions*. In the last years of his short life he started to gain income and fame he

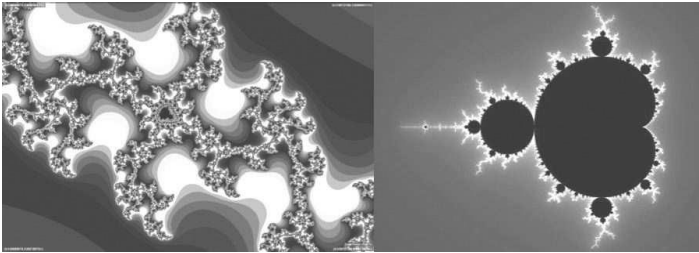
deserve but because of ill-health after one year he returned to India he died on the 26th April 1920, at Chetput, a suburb of Madras.

Another India-related is British mathematician and logician, Alan Turing (1912–1954) born in London on June 23, 1912 with a deep Scottish origin. His parents was working as civil engineers on railway project in India. They abandoned their two sons with an unrelated family in London. After finishing school Alan entered King's College Cambridge in 1931. For his undergraduate dissertation, Turing developed a new proof of the central limit theorem of probability. In his postgraduate work he did his greatest mathematical achievement in 1936 his paper *On Computable Numbers, With an Application to the Entscheidungs problem* i.e. the third question of Hilbert, "to prove the existence of a *decision procedure* to decide the truth or falsity of any given mathematical proposition". To answer, he made exact the notion of a decision procedure, then formalized it. He explained the concept of a decision procedure in terms of machines based on typewriters, which now we called Turing machines. Its aim was obtaining a

mathematical exact definition of what is ‘computable’. The machine could calculate or compute anything using an **algorithm**. During the Second World War, he was involved in cryptanalysis, the breaking of codes, and afterwards worked on the construction of some of the early **digital computers** and the development of their programming systems.

Among the most active pure and applied research field in mathematics today, we have fractals, complexity and chaos. Benoit Mandelbrot, (1924–2010) was a Polish mathematician. He demonstrated the applications of fractals in mathematics and in our nature, with computer graphics. Fractal is a set of points whose fractal dimension is not an integer but a fraction which leads to sets with infinitely complex structure usually using *measure of self-similarity*. By self-similarity measure any part of the set is a reduced scale of the whole set in repeated various structures in different scales. In fact Fractal research begins in 19<sup>th</sup> century. In 1872 Karl Weierstrass explored fractal objects considering functions that were continuous everywhere but not differentiable anywhere. One can visualize

this in graph of noise. Also Helge von Koch, discussed geometric shapes such as Snowflake in 1904. During that time several mathematicians explored fractals in the complex plane, however, they could not fully appreciate or visualize these objects without the aid of the computer. The complexity of the Mandelbrot set (left) is in contrast to the relative simplicity of the entire set (right).

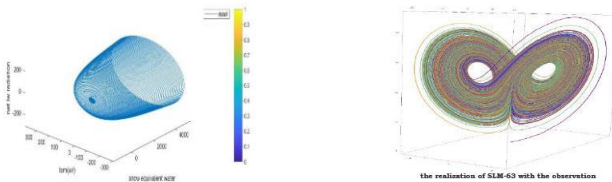


**Chaos** is a situation in which a fully deterministic dynamical process can appear to be random and unpredictable due to the sensitive dependence of the process on its initial values or conditions. For example, iterations of the function such as in fractal structures show chaotic behavior in computer graphics. Lorenz-63 model (attractor) is a famous fractal structure. It arises as the solution to a set of parameterized differential equations

which is reduced form of Naiver–Stokes equations in fluid dynamics as below.

$$\begin{aligned}\frac{dx}{dt} &= \alpha(y - x) \\ \frac{dy}{dt} &= x(\beta - z) - y \\ \frac{dz}{dt} &= xy - \gamma z\end{aligned}$$

These describe the flow of fluid in a convective motion, i.e. heated in bottom and rise and turn back down in repeated circular manner. Edward N. Lorenz used it as a model for the behavior of weather in 1963. Lorenz attractor is a set of chaotic solutions of the Lorenz system which, when plotted, resemble a butterfly or figure eight.



It became one of landmarks in chaos theory. Since the 1960s, individual software packages such as Mathematica have existed for specific numerical, algebraic, graphical, and other tasks,

and researchers interested in chaos and fractals have long used computers for their explorations.

In the end we choose Roger Penrose (1931- ) British, philosopher, mathematician and theoretical physicist. He was Professor of Mathematics at Oxford University for 25 years. He was awarded the Wolf Prize for Physics in 1988 with Stephen Hawking, and Noble prize in physics in 2020. He also collaborated with other pure mathematicians, producing important papers in cosmology, topology, manifolds and twister theory which uses geometry and algebra in an attempt to unite quantum theory and relativity which is the most active research field in physics. He is well known as the author of popular science books such as *Road to Reality*. His works on consciousness, mind and physics is also had revolutionary effects on thinkers and scientists.

*Algorithm* is a set of rules that precisely defines a sequence of operations. This is essential for computers since computer programs contain algorithms that with detail instructions lead a computer to perform a specified task. The traditional computer programs based on crisp numbers, do not support uncertainties. To cope

with this problem A. Lotfi Zadeh an Iranian professor at University of California at Berkeley first proposed fuzzy logic in 1965. Its reasoning power is similar to humans based on concepts of partial memberships. Founded on set theory, Fuzzy logic is able to process incomplete data and provide approximate solutions to problems other methods find difficult to solve.

"In standard set theory an element either is or is not a member of a particular set. However, in some instances, for example in pattern recognition or decision making, it is not known whether or not an element is in the set. Fuzzy set theory blurs this distinction and replaces this two-valued function with a probability distribution giving the likelihood that an element is a member of a particular set".

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## **Math Elements**

In part one we have learned how to think and solve mathematically. In part 2 we have seen prominent mathematician's life story and discoveries. In this part we move to the foundations or elements of mathematics from very familiar and ancient point i.e. numbers, but in its modern point of view. Then goes on to more modern concepts such as sets, logic functions, calculus, differential equations, combinatorics which is a close relative of computation along with the logic, probability and linear algebra. We must keep in mind that modern analogous of old arithmetic-algebra-geometry is computation-combinatorics-logic.

To say any number we need a symbolic shape since number as we said before is an abstract idea. For ancient Egyptian, numbers acquired symbolic meanings in different ways. However Elamits and Babylonians founded place-valued number system base 60. Today we use the same technique but base 10. Generally the way you write down numbers determine that particular number system.

Numbers in elementary school by symbols are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 . . . This symbolism, of base 10. We say 3671. This stands for:

$3000+600+70+1$ , in other words:

$$\begin{aligned} 3671 &= 3 \times 1000 + 6 \times 100 + 7 \times 10 + 1 \\ &= 3 \times 10^3 + 6 \times 10^2 + 7 \times 10 + 1. \end{aligned}$$

Therefore the way one write down numbers is called number system. Decimal system we use today is base 10 system, another example is:

$$434.15 = 4 \times 100 + 3 \times 10 + 4 + 1/10 + 5/100$$

So, our number system is going on with power of 10.

*Factorization* is finding numbers whose product is a given number. 1000 can be gain from  $2 \times 500$ ,  $4 \times 250$ ,  $20 \times 50$ ... and also  $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$  on the other hand  $234^{5678} =$  a very large number, bigger than universe!!

We know *prime* number is a number greater than 1 and not the product of smaller numbers. In fact they are positive integers that are divisible only by themselves and 1. Therefore 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31..... to infinitely. A number that

is neither 1 nor a prime is called a *composite number* which can be written uniquely as a product of prime *factors* multiplied together:  $12 = 2^2 \times 3$ ,  $21 = 3 \times 7$ , and  $270 = 2 \times 3^3 \times 5$ . 15, is the product of two primes,  $15 = 3 \times 5$ , the same is for:

$$13,500 = 3^3 \times 2^2 \times 5^3 = (3 \times 3 \times 3) \times (2 \times 2) \times (5 \times 5 \times 5) = 27 \times 4 \times 125.$$

Again, there is no other way to write these numbers as a product of prime factors. This property is called **fundamental theorem of arithmetic**.

There are rules, or "algorithms" to divide, multiply, add and subtract in base 10 number system which everyone know. However, they have been known since ancient time with other number system such as base 60. Algorithms are fundamental to computer science. An algorithm is a set of rules to solve a problem. The most famous one is the *Euclidean algorithm* in 300 BC for finding the greatest common divisor of two numbers. This allows us to repeatedly remove the larger number in the pair, reducing the size of the numbers involved until one vanishes. The last

nonzero number is then the GCD (Greatest Common divisor) of the original pair.

*Natural numbers* are numbers we can count with our fingers, (0, 1, 2, 3, 4, ...). Infect with counting objects can transform to each other. If addition is summing the objects, multiplication is sums of sums and is the answer of how many objects are in  $a$  groups of  $b$ ? In opposite, division says—if  $b$  objects are divided into  $a$  equal groups, how many objects are in each? Natural numbers will show with

$$N = \{1, 2, 3, \dots\}$$

Thus when we say  $2 \in N$  it means 2 belongs to/is member of a set  $N$  of natural numbers. Sets usually show with capital letters.

When you divide, you make portions or fractions. You say  $3/6=1/2$  or 0.5 while  $6/3=2$ , the former's reciprocal is;  $6 \times 1/2=3$  and the latter's reciprocal is  $3 \times 2=6$ . This is the story of our daily life. In fact fractions are introduced into arithmetic to make division possible. The note is divide 15 into 5 parts:  $15 = 5 + 5 + 5$ . It is not possible to divide, say, 14 into 3 equal parts if we insist that these parts are natural numbers. Therefore we define

fractions as  $m/n$  where  $m, n \in \mathbf{N}$  and  $n \neq 0$  since division by zero is nonsense. To make the process more clear we say,  $2/4$ , here we divide an object into 4 equal pieces and takes 2 of them to get 2 fourths and also we say,  $3/6$ , here we divide the object into 6 equal pieces and take 3 to get 3 sixth. Note is that  $2/4$  and  $3/6$  are *equivalent* since both fractions produced the same results i.e.  $1/2$  and they are in the same point on a *real number* line. Addition and subtraction along with division and multiplication in the set of fractions  $\mathbf{F}$  have some rules. If  $m, n, p, q$  are natural numbers and  $n, q \neq 0$  then:

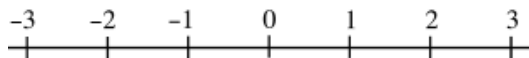
$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq},$$

$$\frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq}.$$

In opposite direction of positive numbers we have negative numbers and both of them are called whole numbers or ***Integers*** ( $\mathbf{Z}$ ).

$$\mathbf{Z} = \{ \dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots \}.$$

On a number line this represents as:



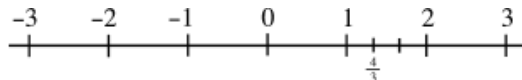
In fact Integers ( $\mathbf{Z}$ ) are introduced into arithmetic to make subtraction possible. The rules are here as if  $a, b$  are two natural numbers then:

$$a - b = c \text{ if } a > b, \quad a - b = -c \text{ if } a < b$$

$a \cdot -b = -c, -a \cdot b = -c$ , and the same is for division. The dot sign means multiplication.

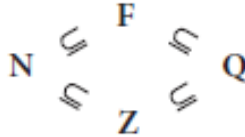
Before we go to *rational numbers*, we introduce another notation of set theory. The symbol  $\subseteq$  means 'is a subset of'. We say  $\mathbf{N} \subseteq \mathbf{Z}$ : The set of natural numbers is a subset of the set of Integers.  $\mathbf{N} \subseteq \mathbf{F}$ : the set of natural numbers is the subset of fractions.

Neither in  $\mathbf{Z}$  and nor in  $\mathbf{F}$  systems operations is not always possible. To make this possible we move into the system of rational numbers  $\mathbf{Q}$  stands for "quotients". For this aim we introduce *negative fractions* into  $\mathbf{F}$  as well. They will place between the integers  $(-, +)$  in a number line.



$\frac{4}{3}$  is placed between *interval* 1 and 2, on one third of 3 *subintervals* between 1 and 2. Arithmetical rules in the set  $\mathbf{Q}$  are the same as the

set  $\mathbf{F}$ , just,  $m, n, p, q$  are integers not natural numbers. To summarize one can show these number system relations as below.



Or in better way as:

### Natural numbers

$$\{1, 2, 3, 4, 5, \dots\}$$

These are the numbers that we use for counting.

### Whole numbers

$$\{0, 1, 2, 3, 4, 5, \dots\}$$

The set of whole numbers includes 0 and the natural numbers.

### Integers

$$\{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$$

The set of integers includes the whole numbers and the negatives of the natural numbers.

## Rational numbers

$$\left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers and } b \neq 0 \right\}$$

The set of rational numbers is the set of all numbers that can be expressed as a quotient of two integers, with the denominator not 0.

Rational numbers can be expressed as terminating or repeating decimals.

$$\begin{aligned} -17 &= \frac{-17}{1}, -5 = \frac{-5}{1}, -3, \\ -2, 0, 2, 3, 5, 17, \\ \frac{2}{5} &= 0.4, \\ \frac{-2}{3} &= -0.6666 \dots = -0.\overline{6} \end{aligned}$$

## Irrational numbers

The set of irrational numbers is the set of all numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient of integers.

$$\begin{aligned} \sqrt{2} &\approx 1.414214 \\ -\sqrt{3} &\approx -1.73205 \\ \pi &\approx 3.142 \\ -\frac{\pi}{2} &\approx -1.571 \end{aligned}$$

In fact irrational numbers (**I**) is a set of real numbers that is not rational. A proof of Pythagoras, shows that

$$\sqrt{2}, \sqrt{3}, \sqrt{7}, 1 + \sqrt{2}, 1/1 + \sqrt{2}, e, \pi \dots$$

Are irrational. In fact their nature makes them not only **infinite**, but **uncountable**. This is also in accordance with the property of **cardinality**. A cardinal number is a number such as 3 or 11 or 412 used to specify quantity but not order (American heritage dictionary). While in the case of a set it is about properties characteristic of infinite sets which are impossible for finite sets.

We do not say "sets *A* and *B* have equally many elements" ; but say: "*A* and *B* have the same *cardinality*" or say: "sets *A* and *B* are *equivalent*." Because of this, the word cardinality means the same thing for infinite sets as the words "number of elements" do for finite sets.

These illustrations sum up the subsets of the set of real numbers **R**. However we can add, *algebraic numbers* which are solutions or roots to polynomial equations with integer coefficients. All rational numbers are algebraic, since  $a/b$  is the root of the equation  $bx - a = 0$ . However there

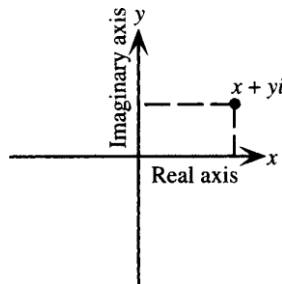
are also irrational numbers which are algebraic, such as  $\sqrt{2}$  is the root/solution of the equation  $x^2 - 2 = 0$ . Those irrational numbers that are not algebraic (such as  $\pi, e, \dots$ ) is called a *transcendental* numbers which are also a set of real numbers that are not roots of a polynomial equation with integer coefficients. But what about negative roots?!

The set of  $\mathbf{R}$  is an extension or generality of the sets  $\mathbf{N}$ ,  $\mathbf{Z}$ ,  $\mathbf{F}$ ,  $\mathbf{Q}$  and  $\mathbf{I}$ . Complex numbers ( $\mathbf{C}$  or  $z$ ) are an extension of the real numbers that make it possible to make sense of the square roots of negative numbers. Any complex number  $z$  can be written as  $a + ib$ , where  $a$  and  $b$  are real and  $i = \sqrt{-1}$  thus  $i^2 = -1$ .  $a$  is the real part of  $z$  and  $b$  is the imaginary part.

Coordinate on a line describe the position of a point on a line or 1- dimensional space. Cartesian coordinate describe the position of a point on a 2-dimensional  $(x, y)$  plane or a 2-D space and also in 3-D space  $(x, y, z)$ . To do this they use a pair and a triple of numbers that describe how to reach that point from an arbitrary origin respectively. In fact one can have  $n$ -dimensional spaces specified

by  $n$  coordinates. With the help of this invention one can have algebraic illustrations to geometrical shapes. In fact this invention is the turning point of classical mathematics toward modern mathematics since then we could have algebraic and differential description of geometry and nature itself. We could have calculus, linear algebra; vectors, Tensors and groups. Thanks to Descartes. With this description we continue to our discussion of complex numbers.

If  $(a,b)$  is a pair of numbers in a  $(x, y)$  Cartesian coordinates, then if we replace  $y$  with  $yi$  then our pair would be  $(a + bi)$ . The plane is called the complex plane when the point  $(x, y)$  is taken to represent the complex number  $x + yi$ . This would result in so-called Argand diagram. The  $x$ -axis in this system is referred to as the *real axis*, and the  $y$ -axis as the *imaginary axis*.



Consequently, any complex number  $z$  as a point in the plane has a distance from the origin, called the modulus (Absolute value) of  $z$  and denoted  $|z|$ , since being negative or positive have no place in a distance. By Pythagoras's theorem  $|z|$  can be calculated from its two components, using

$$|z|^2 = a^2 + b^2 \equiv |z| = r = \sqrt{x^2 + y^2}.$$

"  $\equiv$  " is a symbol for being identical.

Now we move on to logic and set theory as a sound building blocks of modern mathematics. We consider more deeply the previous discussions of primary number sets along their properties. Then we go on to our next fundamental disciplines of mathematics.

Proof, and hence *logic*, is essential to mathematics. Logic is the study of deductive reasoning, by which results are derived from sets of premises (principles). In fact this is the most distinctive feature of mathematics which allow us to prove things as far as there exist no ambiguity. Logical *connectives* are included symbols such as *implies*  $\Rightarrow$ , *there exists*  $\exists$ , *there not exists*  $\nexists$ , and  $\wedge$ , or  $\vee$ , *for all*  $\forall$ , *identical to*  $\equiv$  etc.

From Euclid *elements* we know that, if some elementary statements (premises) are true, then statements constructed from them are also true. *Abstract properties and objects, need accurate formal definition using logical rules.*

As we said there is strong relation between logic and computer programming language. The first basics in computer languages is similar to our daily language. We use different sentences such as *statement* which can be True or False mathematically 1 and 0 in binary form and uncertain. We say "Earthquakes don't happen in Iran" which can be T or F but "The world will end on June 6, 2100" is uncertain. We say our commands with Imperative sentences "wash your teeth before going to bed." With interrogative sentences we say our questions "How much is that painting on the wall?" and when we are surprised we use Exclamations, "What a hot day!" Some time we compound two statements by connectives "my car is broken *and* I have to go with bus."

We use lower case letters such as p, q, r for statements for negative use the negation symbol ( $\neg$ ) read "not" before the letter.

**P**: I rented a house,  $\neg \mathbf{p}$ : I did not rent a house. In a *truth table* we have:

$p$	$\neg p$
T	F
F	T

Given two statements  $p$  and  $q$ , we define the statement  $p \wedge q$  ( $p$  **and**  $q$ ) to be true precisely when both  $p$  and  $q$  are true.

$p \wedge q$  : *she is at least 25 years old, **and** she can live for herself.*

Given two statements  $p$  and  $q$ , we define the statement  $p \vee q$  ( $p$  **or**  $q$ ) to be true precisely when at least one of  $p$  and  $q$  is true.

$p \vee q$  : *she must go with her parents **or** stay at home staring at walls.*

Two statements are said to be *logically equivalent* if they have precisely the same truth table values. If  $p$  and  $q$  are logically equivalent, we write  $p \Leftrightarrow q$ . In fact  $p$  implies  $q$  and  $q$  implies  $p$ . saying;

$p$  **AND** *either  $q$  or  $r$*

Has the same meaning to us as;

$p$  and  $q$ , OR  $p$  and  $r$ .

This is similar to distributive property in real numbers. Statement whose truth table values are all TRUE is called a *tautology* and their negation is called *contradiction*.

The statement  $p \rightarrow q$ , read "If  $p$ , then  $q$ " or " $p$  implies  $q$ " is defined to be a statement that is logically equivalent to  $\neg p \vee q$ . We call  $P$  the *hypothesis* condition and  $q$  the *conclusion*.

While given statements  $p$  and  $q$ , the statement  $p \leftrightarrow q$ , read " $p$  if and only if  $q$ ," or " $p$  iff  $q$ " is defined to be true precisely when  $p$  and  $q$  are either both true and both false.

In the bellow table which statement is stronger,  $p$  or  $p \wedge q$ ?

$p$	$q$	$p \wedge q$	$(p \wedge q) \rightarrow p$	$p \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	T	T

Since  $(p \wedge q) \rightarrow p$  is a tautology while  $p \rightarrow (p \wedge q)$  is not,  $p \wedge q$  is stronger than  $p$ . Knowing  $p \wedge q$

is true guarantees that  $p$  is true, but knowing  $p$  is true does not guarantee that  $p \wedge q$  is true.

To analyze something we say words like ALL and SOME. These words are called *quantifiers*. Logically these two expressions is equivalent with ‘FOR ALL...’  $\forall$  and ‘THERE EXIST... $\exists$ ’ which are called *universal quantifiers*. A phrase such as ‘for all  $x$ ’ or ‘there exists  $x$ ’ can be either true or false. To formalize;

1. For all  $x \in A$ ,  $x \geq \pi$ .
2. There exists  $x \in A$  such that either  $x < \pi$  or  $x > \pi$ .
3. There exists  $x \in N$  such that, for all  $y \in N$ ,  $x \leq y$ .

Into logical statements involving  $\forall$ ,  $\exists$ ,  $\wedge$ , and  $\vee$  we express as bellow:

1.  $(\forall x \in A) (x \geq \pi)$ .
2.  $(\exists x \in A) [(x < \pi) \vee (x > \pi)]$ .
3.  $(\exists x \in N) (\forall y \in N) (x \leq y)$ .

To communicate that exactly one thing with a certain property exists, we say that it exists

*uniquely* ( $\exists!$ ). The statement  $(\exists! x) (P(x))$  "there exist a unique  $x$  such that  $p(x)$  is defined by;

$$[(\exists x) (P(x))] \wedge [(P(x_1) \wedge P(x_2)) \rightarrow (x_1 = x_2)]$$

Now we go on and combine our discussion with set theory to show more rigorous results in mathematics. Modern mathematical courses mostly starts with a set of objects along with measures whether real or abstract. One can build axioms, find and prove their properties using logic as a system called *axiomatic system*. *If our primitive statements proved then we have a new theorem.*

*Set theory* is the study of the properties of sets and their relations, originally developed by Georg Cantor and another German mathematician, Ernst Zermelo (1871-1953) founded *Axiomatic set theory*, against and less ambitious than *set theory* which formalize the nature of a set using a minimal number of independent axioms. *Axiomatic set theory* is a set theory including axioms together with rules of inference.

Suppose  $A$  and  $B$  are sets. We say that  $A$  is a *subset* of  $B$ , written  $A \subseteq B$ , provided the statement "if  $x \in A$ , then  $x \in B$ " is true. That is;

$$(A \subseteq B) \Leftrightarrow (x \in A \rightarrow x \in B) \Leftrightarrow (\forall x \in A) (x \in B).$$

If  $A$  and  $B$  are sets, we say that  $A = B$  provided  $A \subseteq B$  and  $B \subseteq A$ . That is,

$$\begin{aligned} A = B &\Leftrightarrow (A \subseteq B) \wedge (B \subseteq A) \\ &\Leftrightarrow (x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A). \end{aligned}$$

If  $A$  and  $B$  are sets, we define the sets

$$\begin{aligned} A \cup B &= \{x : x \in A \vee x \in B\} \\ A \cap B &= \{x : x \in A \wedge x \in B\}. \end{aligned}$$

$$\begin{aligned} x \in A \cup B &\Leftrightarrow (x \in A \vee x \in B) \\ x \in A \cap B &\Leftrightarrow (x \in A \wedge x \in B). \end{aligned}$$

If  $A \cap B = \emptyset$ , then  $A$  and  $B$  are said to be *disjoint*. To say  $A \cap B = \emptyset$  is to say that there does not exist any  $x \in A \cap B$ . That is,

$$\begin{aligned} A \cap B = \emptyset &\Leftrightarrow \neg(\exists x \in A \cap B) \\ &\Leftrightarrow \neg(\exists x \in A)(x \in B) \\ &\Leftrightarrow (\forall x \in A)(x \notin B) \\ &\Leftrightarrow x \in A \rightarrow x \notin B. \end{aligned}$$

If  $A$  and  $B$  are sets, we define the *difference* of  $A$  and  $B$  as

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\} = A \cap B'.$$

If  $A$  and  $B$  are sets, we define the *symmetric difference* of  $A$  and  $B$  as

$$\begin{aligned} A \Delta B &= \{x : x \in A \cap B' \text{ or } x \in A' \cap B\} \\ &= (A \cap B') \cup (A' \cap B). \end{aligned}$$

If  $A$  and  $B$  are sets, then

$$(A \cap B)' = A' \cup B'.$$

This is De Morgan's law. For our discussion of statements it would be as,

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q).$$

Which is a tautology.

Instead of saying "a set of sets" we say a *family* or *collection* of sets. We could write a family of  $n$  sets where  $N_n$  is an *index set* for the family of sets as,

$$\{A_1, A_2, A_3, \dots, A_n\} = \{A_k : k \in N_n\} = \{A_k\}_{k=1}^n,$$

This would help us to generalize intersection and union as bellow. ( $k \in N_n$ )

$$A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = \bigcup_{k=1}^n A_k$$

$$A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n = \bigcap_{k=1}^n A_k.$$

We conclude this part with **algebra of sets**.

The set of all subsets of a universal set  $E$  is closed under the binary operations  $\cup$  (union) and  $\cap$  (intersection) and the unary operation  $'$  (complementation). If  $A, B$  and  $C \subseteq$  of  $E$  then we have.

1  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  and  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ , (**associative properties**)

2  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ , (**commutative properties**)

3  $A \cup \emptyset = A$  and  $A \cap \emptyset = \emptyset$ , where  $\emptyset$  is the empty set.

4  $A \cup E = E$  and  $A \cap E = A$ .

5  $A \cup A = A$  and  $A \cap A = A$ .

6  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ , (**distributive properties**)

$$7 \ A \cup A' = E \text{ and } A \cap A' = \emptyset.$$

$$8 \ E' = \emptyset \text{ and } \emptyset' = E.$$

$$9 \ (A')' = A.$$

10  $(A \cup B)' = A' \cap B'$  and  $(A \cap B)' = A' \cup B'$ , the last one is De Morgan's laws we mentioned above. This law in universal notation we introduced would be as: Suppose  $\mathcal{F}$  is a family of sets. Then

$$\left[ \bigcup_{\mathcal{F}} A \right]' = \bigcap_{\mathcal{F}} A'.$$

The properties about the sets above have their equivalents in the set of real numbers  $\mathbf{R}$ . Thus we also have basic algebraic properties of real numbers which are extremely important in our coming discussions. These properties originated from 10 formal *axioms* of  $\mathbf{R}$  which will come later.

*1 For all  $a, b, c \in \mathbf{R}$ , if  $a + c = b + c$ , then  $a = b$ .*

*2 For every  $a \in \mathbf{R}$ ,  $a \cdot 0 = 0$ .*

*3 The additive inverse of a real number is unique.*

**4** For every  $a \in R$ ,  $-(-a) = a$ .

**5**  $-0 = 0$ .

**6** If  $a, b \in R$ , then;

$$1. (-a)b = -(ab),$$

$$2. (-a)(-b) = ab.$$

**7** If  $b \in R$ , then;

$$(-1)b = -b. \quad \text{or} \quad -(a+b) = (-1)(a+b) = (-1)a + (-1)b = (-a) + (-b).$$

*(Distributive properties)*

**8** If  $a, b \in R$ , then  $-(a+b) = (-a) + (-b)$ .

This means "the additive inverse of a sum is the sum of the additive inverses".

**9** If  $ac = bc$  and  $c \neq 0$ , then  $a = b$ .

**10** The multiplicative inverse of  $a \neq 0$  is unique.

**11** For all  $a \neq 0$ ,  $(a^{-1})^{-1} = a$

Note and proof is that:

$$a^{-1} = \frac{1}{a}$$

Thus

$$(a^{-1})^{-1} = \left(\frac{1}{a}\right)^{-1} = \frac{1}{\frac{1}{a}} = 1 \div \frac{1}{a} = 1 \times \frac{a}{1} = \frac{a}{1} = a$$

This means inverse of inverse term you back the original value.

**12** For all nonzero  $a, b \in \mathbf{R}$ ,  $(a b)^{-1} = a^{-1} b^{-1}$ .

**13** For all nonzero  $a \in \mathbf{R}$ ,  $(-a)^{-1} = -(a^{-1})$ .

Note and proof is that:

Suppose  $a \neq 0$ . Then there exists  $a^{-1} \in \mathbf{R}$ , and by part 2 of Theorem 6 we have:

$$1 = a \cdot a^{-1} = (-a) [-(a^{-1})]$$

Since  $a \neq 0$ , this is true for  $-a$  and  $a = -(-a) = 0$  by theorem 5. Thus there exists  $(-a)^{-1} \in \mathbf{R}$ .

Now we consider real numbers. It is an ordered set. This means order is important in  $\mathbf{R}$ . This leads to commutative property and symmetric operations. We say:

If  $a > 0$ , we call  $a$  *positive* ( $\mathbf{R}^+$ ), and if  $0 > a$  (or  $a < 0$ ), we call  $a$  *negative* ( $\mathbf{R}^-$ ).  $\mathbf{R}^-$ ,  $\{0\}$  and  $\mathbf{R}^+$  are parts of  $\mathbf{R}$  which doing symmetrical and commutative operations possible. Theses leads to properties for inequalities as below.

**1** If  $a > b$ , then  $-a < -b$ .

**2** if  $c \in \mathbb{R}$ . Then  $c > 0$  if and only if  $-c < 0$ .

Absolute value is a very important measure of the *size* of a real number.

**3** For  $x \in \mathbb{R}$ , we define  $|x|$ , the *absolute value* of  $x$  by

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

This means:

$$1 \quad |ab| = |a||b|.$$

$$2 \quad |a + b| \leq |a| + |b|.$$

$$3 \quad |a - b| \geq ||a| - |b||.$$

4 For  $a > 0$ ,  $|x| \leq a$  if and only if  $-a \leq x \leq a$ .

**4** If  $a \geq 0$ . Then  $|x| < a$  if and only if  $-a < x < a$ .

**5** if  $a \geq 0$ . Then  $|x| > a$  if and only if either  $x > a$  or  $x < -a$ .

**6** For all  $x, y \in \mathbf{R}$   $|x + y| \leq |x| + |y|$ . This is called Triangle inequality. However for all  $x, y \in \mathbf{R}$ ,  $|x - y| \geq |x| - |y|$  and  $||x| - |y|| \leq |x - y|$  are also true and triangular.

As we said earlier the set  $\mathbf{C}$  of complex numbers  $(a + bi)$  comes from the fact that there is no real number  $x$  such that  $x^2 + 1 = 0$ . If  $b = 0$  then  $\mathbf{C}$  includes all the real numbers as well. It is assumed that two such numbers may be added and multiplied using the familiar rules of algebra, with  $i^2$  replaced by  $-1$  whenever it occurs such as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Note is that the set  $\mathbf{C}$  is not an ordered set.

It is important to remember that, introducing rational numbers ( $\mathbf{I}$ ) into  $\mathbf{N}$  and  $\mathbf{Q}$  gave us  $\mathbf{R}$ . Thus let's consider more about it. Properties of set  $\mathbf{I}$  are: it is infinite, ordered (smaller or greater), and dense everywhere, this means that between any different element of  $\mathbf{I}$  there exist infinite elements of  $\mathbf{I}$  as well. However every rational number can be represented as a terminating or periodically

infinite *decimal fraction* or as a finite *continued fraction*.

The set of rational numbers is not satisfactory for calculus. Even though it is dense everywhere, it does not cover the whole numerical axis. The introduction of irrational numbers allows us to assign a number to every point of the numerical axis. Irrational numbers take all the non-rational points of the numerical axis. They correspond to a point of the axis and can be represented as a non-periodic infinite decimal fraction.

To sum up we say that; the set of real numbers  $\mathbf{R}$  equipped with two binary operations of addition and multiplication, has 13 axioms to construct and prove theorems of *elementary* and *abstract* algebra. Below the lower-case letters such as  $a$ ,  $b$ ,  $c$  or  $x$ ,  $y$ ,  $z$  represent arbitrary real numbers. They grouped in *field axioms*, *order axioms*, and *completeness axiom* (least-upper bound axiom or continuity axiom).

Let  $\mathbf{R}$  be a set, equipped with two binary operations, addition and multiplication. If  $x, y, z \in \mathbf{R}$  and  $x + y$  the *sum* of  $x$  and  $y$  and  $x.y$  the *product* then:

Axiom 1. Commutative laws.

$$x + y = y + x, \quad xy = yx.$$

Axiom 2. Associative laws.

$$x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z.$$

Axiom 3. Distributive law

$$x(y + z) = xy + xz.$$

Axiom 4. Existence of identity elements.

For every real  $x$  we have;

$$x + 0 = x \text{ and } 1 \cdot x = x.$$

Axiom 5. Existence of negatives.

For every real number  $x$  there is  
a real number  $y$  such that;

$$x + y = 0.$$

or given  $a \in \mathbf{R}$  there exists

$$-a \in \mathbf{R} \text{ such that } a + (-a) = 0.$$

Axiom 6. Existence of reciprocals.

For every real number  $x \neq 0$  there is a real number  $y$   
such that  $xy = 1$ . Or given  $a \in \mathbf{R}$ ,  $a \neq 0$ , there exists  $a^{-1} \in \mathbf{R}$   
such that  $aa^{-1} = 1$ .

From the above axioms we can deduce all the usual arithmetic laws of elementary algebra said earlier. In fact they are field axioms of  $\mathbf{R}$ . A set  $\mathbf{R}$  with binary operations  $+$  and  $\times$  is said to be a **field** if for all  $x, y, z \in \mathbf{R}$ , if all above six axioms is satisfied.

Ordering enables us to have one real number being larger or smaller than another. The usual order relation is:

$$a \geq b \Leftrightarrow a - b \in \mathbb{R}^+$$

A field  $\mathbf{R}$  is **ordered** if there exists a subset  $\mathbf{R}^+ \subseteq \mathbf{R}$  such that:

Axiom 7. If  $x, y \in \mathbf{R}^+ \Rightarrow x + y, xy \in \mathbf{R}^+$

Axiom 8.  $x \in \mathbf{R} \Rightarrow x \in \mathbf{R}^+$  or  $-x \in \mathbf{R}^+$ . Or for every real  $x \neq 0$ , either  $x \in \mathbf{R}^+$  **OR**  $-x \in \mathbf{R}^+$ , but not both.

Axiom 9.  $0 \notin \mathbf{R}^+$ . Or  $(a \in \mathbf{R}^+) \mathbf{AND} (-a \in \mathbf{R}^+) \Rightarrow a = 0$

The set  $\mathbf{R}$  is an intuitive concept. An element  $x \in \mathbf{R}$  is an *upper bound* for  $S \subseteq \mathbf{R}$ , if  $x \geq s$  for all  $s \in S$ . A set  $S$  with an upper bound is said to be *bounded above*. An element  $e$  of  $\mathbf{R}$  is a *least upper bound* for  $S$  if:

$e \geq s$  for all  $s \in S$  ( $e$  is an upper bound) and  $x \geq s$  for all  $s \in S \Rightarrow x \geq e$  ( $e$  is the least among the upper bounds).

Axiom 10. **Completeness/ LUB**

If  $S$  is a non-empty subset of  $\mathbf{R}$  and  $S$  is bounded above, then  $S$  has a least upper bound in  $\mathbf{R}$ . Or every nonempty set  $S \subseteq \mathbf{R}$  which is bounded above has a supremum; that is, there is a real number  $e$  such that  $e = \sup S$ .

**Elementary Algebra** is the area of mathematics related to the general properties of arithmetic. Its relationships can be summarized by using variables, usually denoted by letters  $x, y, n, \dots$  to stand for unknown quantities, whose values may be determined by solving the resulting *equations*.

As a rule, in all previous cases of number sets it might be noted that the theories would hold true completely if we restricted ourselves within the room of the *rational numbers*. The reason for this **parallelism** is an existed generality in algebraic language. This fact provide us the concept of a **field**, and also in the broader sense the concept of a **ring** and thus **Abstract algebra**.

Evidently, the systems of all **C**, **R** and **Q**, like the system of all **Z**, have one property in common: *"they are all **closed** not only under addition and multiplication, but under subtraction as well"*.

Doing operations of sum, difference and product in any system of numbers, such as **C** or **R**, with any two of its numbers is termed as *number ring*. Thus, the systems of all **Z**, **Q**, **R** and **C** numbers are number rings. However; "no system of positive numbers is a ring since if  $a$  and  $b$  are two

different numbers, then either  $a - b$ , or  $b - a$  is negative. Neither is a system of negative numbers a ring because the product of two negative numbers is positive".

Thus "the set of even numbers is a ring; generally, for any natural number  $n$  the collection of integers exactly divisible by  $n$  is a ring. The odd numbers do not constitute a ring since the sum of two odd numbers is an even number".

A number ring is called a **number field** if it contains the quotient  $\left(\frac{p}{q}\right)$ ,  $q \neq 0$  of any two of its numbers. Thus we have the field of **Q**, the field of **R**, and the field of **C**. While the ring of integers does not establish a field.

*"The field of rational numbers lies entirely within any number field".*

Why? Since there exist some number field  $P$  and if  $a \in P$ ,  $a \neq 0$ , then  $P$  also contains the quotient  $\frac{a}{a} = 1$ . Now successive adding of unity to itself, we find all **N** lie in the field  $P$ . however,  $P$  contains the difference  $a - a = 0$ . Thus  $P$  contains the result of subtracting any natural number from zero, i.e. any negative

integer. And in conclusion, the field  $P$  contains the quotients of all integers, or, generally, all rational numbers.

As we mentioned above is a *Ring*. A *Ring* includes a set of elements together with *two* binary operations. However like group which comes later the results of operations existed as another element of the same set and then of the Ring. Formally a ring is:

A set  $R$  with *two* binary operations  $+, *$  is called a ring  $(R, +, *)$  if:

$(R, +)$  is an Abelian (commutative) Group,  $(R, *)$  is a semi-group (satisfy 2 & 4 Axioms) and the Axiom 3 is hold as well (of the first definition of the group).

If  $(R, *)$  is commutative or if  $(R, *)$  has a neutral element, then  $(R, *)$  is called a *commutative ring* or a ring with *identity ring* with unit element respectively. A ring is called a field if  $(R, *)$  is an Abelian group. Therefore, every field is a special commutative ring with identity.

A Field is a set  $F$  which is closed under two operations addition and multiplication with the following properties:

- (1)  $F$  is an abelian group under addition and
- (2)  $F - \{0\}$  i.e. the set  $F$  without the additive identity  $0$  is an abelian group under multiplication.

In fact these operations are examples of **algebraic operation**. The general definition of algebraic operation is as follows.

An *algebraic operation* is defined on the set  $M$  if for any two elements  $a, b$  of the set  $M$  there exist a uniquely associated third element  $c$  which also belongs to  $M$ . This operations may be *addition/sum, multiplication/product* of the elements  $a$  and  $b$  or any new terminology and symbolism introduced for an operation defined on  $M$ . for instance "the collection of functions defined for all real  $x$  becomes a ring when we introduce the operations of addition and multiplication. Thus we could have a **ring of functions** as well. And this is natural since a number is a position holder in space  $p(x)$ .

Remember, both in number system and in the system of polynomials, the algebraic operations have zero, inverse, distributive, associative and commutative properties. This also true for rings.

In the case of the product of  $n$  equal elements; we have the concept of a *power*,  $a^n$ , of the element  $a$  with positive integer exponent  $n$ . all the ordinary rules for operating with exponents hold true in any *ring*. Also in the same way associative law of addition leads to the concept of a *multiple*,  $na$ , of the element  $a$  by a positive integer coefficient  $n$ .

**Abstract algebra** includes Group theory and linear algebra with its matrix theory. In fact elementary algebra deal with simple algebraic expressions, while abstract algebra is the theory of **algebraic structures** that is an abstract concept defined as consisting of certain elements with operations satisfying above given particular 10 axioms of real numbers **R** with some restrictions. A **group**, a **ring** and a **field** are examples of an algebraic structure. A structure satisfying all 10 axioms of **R**, is called a **complete ordered field**.

Generally and roughly Groups, rings, and fields construct by various ways of combining elements to produce an element of the set. In fact they includes sets of elements with additional structure.

Informally a Group  $(G, *)$  is a set in which you can perform **one operation** (usually addition or multiplication) with some special properties. A Ring  $(R, +, *)$  is a set equipped with **two operations**, called addition and multiplication. Thus a *ring* is a *group* under addition and satisfies some of the properties of a group for multiplication. A Field  $(F, +, *)$  is a *group* under both addition and multiplication.

Thus a Group  $(G, *)$  is a set of elements together with a *binary operation*. A binary operation on  $A$  is a function from  $A \times A$  to  $A$ . The set of real numbers with addition and the set of non-zero real numbers with multiplication, are examples of groups. Formally we define a group  $(G, *)$  as:

For any set  $G$ , any binary operation  $(*)$  and three elements  $a$ ,  $b$ , and  $c$ , four basic properties or axioms must be satisfied:

1. *Closure*: if  $a$  and  $b$  are in  $G$ , then so is  $a * b$
2. *Associativity*:  $a (b * c) = (a * b) c$
3. *Identity*: there exists an element  $e$  in  $G$  such that  $e * a = a$  for all  $a$  in  $G$ .
4. *Inverse*: for all  $a$  in  $G$  there exists  $a^{-1}$  in  $G$ , such that  $a * a^{-1} = e$ , where  $a^{-1}$  is the inverse element of  $a$ .

Or in another definition;

A group is a set  $G$  which is closed under one operation  $(*) \exists x, y \in G: x * y \in G$  and satisfies the following properties:

- (1) *Identity*; there exist an element  $e \in G : \forall e * x = x * e = x$  is hold.
- (2) *Inverse*;  $\forall x \in G \exists y \in G: x * y = y * x = e$  where  $e$  is an identity element.
- (3) *Associativity*; The following identity holds  $\forall x, y, z \in G: x * (y * z) = (x * y) * z$ .

We have a *finite group* if it consists of a finite number of elements; if not it is an *infinite group*. The number of elements of a finite group is called its *order*.

**Symmetric group** consider different ways that a structure can be transformed with different combinations by applying one transformation to the result of another. Therefore, the results of

these operations of *composition* would be rotations/spins with different combinations. Formally we define *symmetric group* as:

For any set  $X$ , a permutation of  $X$  is a one-to-one onto mapping from  $X$  to  $X$ . If  $X$  has  $n$  elements, there are  $n!$  Permutations of  $X$  and the set of all of these, with composition of mappings as the operation, forms a group called the *symmetric group of degree  $n$* , denoted by  $S_n$ . This is based on a binary relation on a set  $S$  which is symmetric if, for all  $a$  and  $b$  in  $S$ , whenever  $a * b$  then  $b * a$  i.e. a *commutative* relation.

The famous examples of group structures are as below;

1. *The group of whole numbers.*
2. *The group of rotations of an equilateral triangle (a cyclic group of order 3).*
3. *Klein's four-group.*
4. *The group of rotations of a square (a cyclic group of order 4).*

For instance, all possible rotations of an *equilateral triangle  $abc$*  about its centroid  $e$  is a

group of symmetries of elements,  $\{e, a, a^2, b, ab, a^2b\}$  where  $a$  is a *rotation* of  $120^\circ$  around the center and  $b$  is a *reflection* which has two obvious nontrivial subgroups, the *rotations*  $\{e, a, a^2\}$ , and the *reflections*  $\{e, b\}$ . If we denote the zero rotation by  $a_0$ , the rotation through  $120^\circ$  by  $a_1$  the rotation through  $240^\circ$  by  $a_2$ , then we obtain the following required relations based on required properties we mentioned.

$$\begin{aligned}
 a_0 + a_0 &= a_0 \\
 a_0 + a_1 &= a_1 + a_0 = a_1 \\
 a_0 + a_2 &= a_2 + a_0 = a_2 \\
 a_1 + a_1 &= a_2 \\
 a_1 + a_2 &= a_2 + a_1 = a_0 \\
 a_2 + a_2 &= a_1
 \end{aligned}$$

The addition table of this group would be represented as below. Where we find the sum of two elements at the point of intersection of the row corresponding to the first element with the column corresponding to the second element.

	$a_0$	$a_1$	$a_2$
$a_0$	$a_0$	$a_1$	$a_2$
$a_1$	$a_1$	$a_2$	$a_0$
$a_2$	$a_2$	$a_0$	$a_1$

This is an example of *cyclic group* since a group  $G$  is cyclic if there is an element  $a$  in  $G$  such that the subgroup generated by  $a$  is the whole of  $G$ . *Simple groups* include the cyclic groups of prime order and the family of alternating groups, originate from finite sets. There are 16 *Lie-type groups*, 26 *sporadic groups* with special cases with 20 *Monster groups* and 6 *pariahs*.

Elements of **Lie groups** depend on continuous variables thus they are continuous structures such as a circle with *continuous parameterization*. They do not have, discrete structures of the Monster group and the symmetry groups of polygons. Because of their continuous and smooth structure they are **differentiable manifolds**, which are specific types of topological spaces. Topological spaces are collection/ family of open sets  $\mathbf{T}$  with every point

sufficiently close to any point in the set is also in the set. The set  $T$  must contain  $\{\emptyset\}$ ,  $\cap$  and  $\cup$  between and with its subsets.

Now we finish our discussion of abstract algebra and go on with the most important concept, i.e. function. A function arises from concept of "relation", "Cartesian products" and "ordered pairs".

In a *relation* two things are referred to each other, derives from notation of equality. Logically this relation can be true or not. For instance, the statement  $a > b$ ,  $a = b$ ,  $a \in A \dots$  where  $a$  and  $b$  are integers, are relations that can be either true or not true. Relations can occur between elements in different sets; that is, we can have a relation between elements in a set  $A$  and those in a set  $B$ . in fact A *relation* is a *set construction* that puts all kinds of element comparisons such as ( $\leq$ ), divisibility, and subset inclusion into one mathematical idea. It's also a way of linking elements of two different sets together, and is a context in which functions can be defined. Formally we define a relation as:

Let  $A$  and  $B$  be sets then a *relation between  $A$  and  $B$*  is a *subset  $R$  of Cartesian product* between the set  $A$  and the set  $B$  i.e.  $R$

$\subseteq A \times B$ . for instance,

$A = \{1, 2, 3\}$  and  $B = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ ,  
and

$R = \{(1, 11), (2, 21), (2, 22), (3, 31), (3, 32), (3, 33)\}$

We have a *relation* from  $A$  to  $B$ .

If  $A = B$ , a *relation on  $A$* , is a subset of  $A \times A$ . Therefore generally, a *relation* ( $\sim$ ) on a set is as a subset  $R$  of the Cartesian product  $S \times S$ , for a given ordered pair  $(a, b) : (a, b) \in R$  or  $(a, b) \notin R$ . An *ordered pair* involves two objects in a particular order. Thus, if  $a \neq b$ , then  $(a, b) \neq (b, a)$ .

If *relation* derives from notation of equality, and  $S$  be a nonempty set, then “ $x \equiv y$ ” for all  $x, y \in S$  is said to be an *equivalence relation* on  $S$  if these properties be true for all elements of  $S$ .

1  $x \equiv x$  for all  $x \in S$  (Reflexive);

2 If  $x \equiv y$ , then  $y \equiv x$  (Symmetric);

3 If  $x \equiv y$  and  $y \equiv z$ , then  $x \equiv z$  (Transitive).

Now If  $R \subseteq A \times A$  has the following properties then  $R$  has *equivalence relation on A*.

- 1 If  $(x \sim x) \in R$  for all  $x \in A$  (*Reflexive*)
- 2 If  $(x \sim y) \in R$ , then  $(y \sim x) \in R$  (*Symmetric*)
- 3 If  $(x \sim y), (y \sim z) \in R$ , then  $(x \sim z) \in R$  (*Transitive*).

These two are the same with different notations. Equivalence relation is a very powerful and fundamental idea especially in analysis. It separate  $S$  into nonempty and non-overlapping ( $A \cap B = \emptyset$ ) subsets called *partitions*. Only elements of the same partition or subset are all equivalent to each other that is called an *equivalence class*. Formally we define partition as:

If  $S$  is a set, and  $F = \{A\}$  is a family of subsets of  $S$ . Then  $F$  is said to be a *partition* of  $S$  provided that;

- 1  $A \neq \emptyset$  for all  $A \in F$ ;
- 2 If  $A, B \in F$ , and  $A \cap B = \emptyset$ , then  $A = B$ ;
- 3  $\bigcup_{A \in F} A = S$

If there exist an equivalence relation ( $\equiv$ ) on a nonempty set  $S$  then for an element  $x \in S$ , we define;

$$[x] = \{y \in S : y \equiv x\}.$$

$[x]$  is the set of all elements of  $S$  that are equivalent to  $x$ . This subset of  $S$  is called the *equivalence class* of  $x$ . consequently;

If there exist equivalence relation ( $\equiv$ ) on a set  $S$  and  $F = \{[x] : x \in S\}$  is, the family of all equivalence classes in  $S$ . Then  $F$  is a *partition* of  $S$ .

There is another type of important and fundamental relation called *order relation* which arises from existence of order between the numbers. For instance if  $x \in R$  then,

$$4 < 5, \quad 7 > 2\pi, \quad x^2 \geq 0, \quad 1 - x^2 \leq 1$$

We know that:

$x < y$  means the same as  $y > x$ ,

$x \leq y$  means the same as  $y \geq x$ ,

$x \leq y$  means the same as  $x < y$  or  $x = y$ ,

$x < y$  means the same as  $x \leq y$  and  $x \neq y$ .

Therefore,  $<, >, \leq, \geq$  are all connected with each other.

If  $R (\sim)$  is a relation on a set  $A$  with the following properties:

- 1)  $x \sim x$  for all  $x \in A$  (Reflexive);
- 2) If  $x \sim y$  and  $y \sim x$ , then  $x = y$  (Anti-symmetric);
- 3) If  $x \sim y$  and  $y \sim z$ , then  $x \sim z$  (Transitive).

Then  $R$  is called an *order relation* on  $A$ , or a *partial ordering* of  $A$ .

There is two kind of order relation *weak order* ( $\leq, \geq$ ) and *strong relation* ( $<, >$ )

If we define the *Cartesian product* of two sets  $A$  and  $B$  as:  $A \times B = \{(a, b) : a \in A, b \in B\}$ , we have a set of **ordered pairs** where the first term is an element of  $A$  and the second is an element of  $B$ . The Cartesian product represented as  $\mathbf{R} \times \mathbf{R}$  ( $\mathbf{R}^2$ ) can be extended to  $\mathbf{R} \times \mathbf{R} \times \mathbf{R} \dots (\mathbf{R}^n)$ .

The *Cartesian product*  $A \times B$ , of sets  $A$  and  $B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . Similarly, the Cartesian product  $A \times B \times C$  of sets  $A$ ,  $B$  and  $C$  can be defined as the set of all ordered triples  $(a, b, c)$ , where  $a \in A$ ,  $b \in B$  and  $c \in C$ . More generally, for sets  $A_1, A_2 \dots A_n$ , the Cartesian product  $A_1 \times A_2 \times \dots \times A_n$  can be defined in a similar way.

We can represent a relation graphically, as a set of points in the 2-D or  $xy$ -plane i.e. Cartesian or rectangular coordinates. Because, *functions* represent relationships between variables as *input/ domain/ ordinate* and *output/ range/ codomain/ coordinate*, they take input, operate rule or relationship and produce output such as a *machine*. Function is defined based on set notations and relations. Formally we define a function as:

A function  $f$  from  $A$  to  $B$  is a rule that assigns to each  $a \in A$  a unique element  $f(a) \in B$ .  $A$  is the domain of  $f$  and  $B$  is the codomain of  $f$  i.e.  $f : A \rightarrow B$ . Therefore;

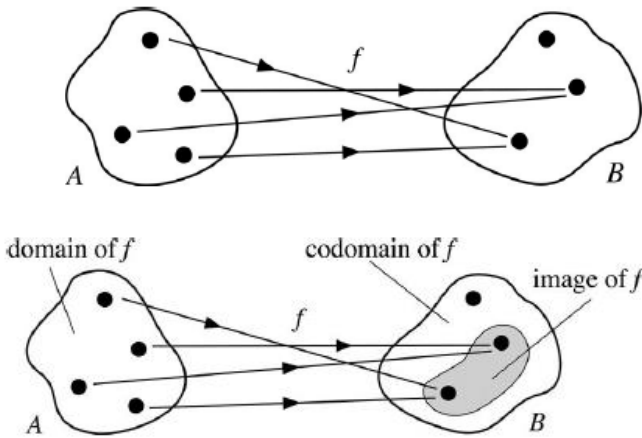
Let  $A$  and  $B$  be sets. A function  $f : A \rightarrow B$  is subset  $f$  of  $A \times B$  such that:

**1** If  $x \in A \exists y \in B \Rightarrow (x, y) \in f$ .

**2** If  $y$  is unique i.e. if  $x \in A$  and  $y, z \in B \Rightarrow (x, y) \in f$  and  $(x, z) \in f$ , then it follows that;  $y = z$ .

Since  $x$  ranges over some set of possible choices, and so does  $f(x)$ , this is also a **mapping** process. Because the result of Cartesian product is ordered

pairs of; (domain, range)/ (domain, codomain)/ (pre-image, image)/ (input, output)/ (ordinate, coordinate) and we use the term *mapping* to refer to any pairing of the elements of two sets. Pictorially it shows:



The second figure shows a mapping process of the function  $f$  which chooses all values of domain space (set or interval) as a pre-image of the function, operate a relation then gives particular values of codomain space (set or interval) as an image of the function.

Mapping is a generalization of function. If we consider a mapping (function)  $f$  as a binary relation between A and B;  $(f \subseteq A \times B)$  is a mapping from A to B provided that:

1.  $\forall a \in A \exists b \in B : ((a, b) \in f)$
2.  $\forall a \in A, \forall b_1, b_2 \in B : ((a, b_1), (a, b_2) \in f \Rightarrow b_1 = b_2)$

The sign ":" reads as *such that*.

Or simply a function  $f$  from  $A$  to  $B$  also is a mapping of  $A$  to  $B$ . We write  $f: A \rightarrow B$ , and this is read " $f$  maps  $A$  to  $B$ " or " $f$  is a function from  $A$  to  $B$ ." The domain of  $f$  is the set  $A$  and the set  $B$  is called the codomain or range of  $f$ .

In addition to these:

$f$  is called a *one-to-one* -or *injective* mapping if:

$$\forall a_1, a_2 \in A, \forall b \in B ((a_1, b), (a_2, b) \in f \Rightarrow a_1 = a_2)$$

Or simply a function  $f: A \rightarrow B$  is an *injection*, or *one-to-one*, if for all  $a_1, a_2 \in A$ ,  $f(a_1) = f(a_2)$  implies  $a_1 = a_2$ . In injective mapping, in terms of on choice of domain there is only one choice of range. For instance:

$f: \mathbf{R} \setminus \{0\} \rightarrow \mathbf{R}$  where  $f(x) = 1/x$ . This is an injection, since if  $1/x = 1/y$  then  $x = y$ .

$f$  is called a *surjective* mapping from  $A$  onto  $B$  if:

$$\forall b \in B \exists a \in A : ((a, b) \in f)$$

Or simply a function  $f: A \rightarrow B$  is a *surjection* to  $B$  or is *onto*  $B$  if each element of  $B$  is of the form  $f(a)$  for some  $a \in A$ . here the codomain is cleared in advanced. For instance:

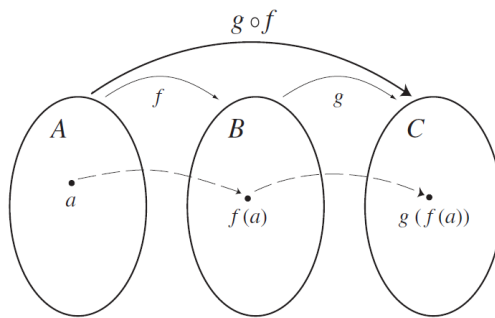
$f: \mathbf{R} \rightarrow \mathbf{R}^+$  where  $f(x) = x^2$ . This is a surjection to  $\mathbf{R}^+$ , since every positive real number has a square root, which is real.

An *injective* mapping that is also *surjective* is called *bijective*. In other words, a mapping that is *injective* or *one-to-one* and *surjective* or *onto*. For a *bijective* mapping/function  $f: A \rightarrow B$ , there is an *inverse relation* leads to an inverse mapping /function  $f^{-1}: B \rightarrow A$  which also is used for composition of functions. Composition of functions is defined as:

If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are mappings, then  $f \circ g$  is also a mapping from  $A$  to  $C$  and expressed as:

$$(f \circ g)(a) = g(f(a)) \quad \text{or} \quad g \circ f : A \rightarrow D.$$

Pictorially it shows as:



For instance: if  $f : R \rightarrow R^+ \cup \{0\}$  defined by  $f(x) = x^2$  and  $g: R^+ \cup \{0\} \rightarrow R^+ \cup \{0\}$  defined by  $g(x) = \sqrt{x}$ . Thus  $(g \circ f)(x) = \sqrt{x^2} = |x|$ .

A mapping/function, which assigns to every element from an *abstract space*  $X$  a unique element usually from a different *abstract space*  $Y$  is called an **operator**. This abstract space is also called a **function space** such as *linear spaces*, *vector spaces*, *metric spaces* and *normed spaces*

However in a general and more useful way we can define a function as:

If  $x$  and  $y$  are two variable quantities, and if there is a rule which assigns a unique value of  $y$  to a given value of  $x$ , then we call  $y$  a function of  $x$  and we use the notation:

$$y = f(x)$$

The variable  $x$  is called the *independent* variable or the *argument* of the function  $y$ . The values of  $x$  to which a value of  $y$  is assigned form the domain *Dom* of the function  $f(x)$ . The variable  $y$  is called the *dependent* variable; the values of  $y$  form the range *Rng* of the function  $f(x)$ . Thus

functions can be represented by the points  $(x, y)$  as curves or graphs.

If both the domain and the range contain only real numbers  $\mathbf{R}$  we call the function  $y = f(x)$  a **Real function** of real variables. For example:

$$y = x^2$$

with Dom:  $-\infty < x < \infty$ , Rng:  $0 \leq y \leq \infty$

$$y = \sqrt{x}$$

with Dom:  $0 \leq x \leq \infty$ , Rng:  $0 \leq y < \infty$

Similar to real functions, we can have complex functions which assign complex of Dom to values to complex values of Rng. Therefore, a function involving complex variables as either input or output, but usually both is a complex function. Then the function  $w = f(z)$  is a mapping from the complex  $z$  plane to the complex  $w$  plane. If the variable  $y$  depends on several independent variables  $x_1, x_2, x_3, x_4 \dots x_n$  then we have a multivariable function.

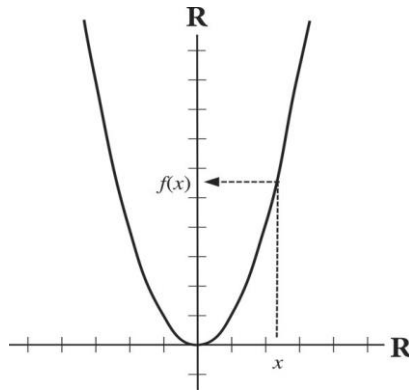
$$f(x_1, x_2, x_3, x_4 \dots x_n)$$

The notions of limit, continuity, and derivative of a real function  $y = f(x)$  can be defined. This is also possible for functions of complex variables or complex functions and multivariable function as well. In fact we have some general definitions and rules for limit, integration and differentiation, then according to different types of **real valued functions** and **complex valued functions** the shape and appearance of those rules will change but the essence remain unchanged. In multivariable function those rules are defined for several variables or parameters which deal with **multivariable calculus** against **single variable calculus**. If you add direction to these scalar values then you have **vector valued functions**. For now we go on with typical different types of functions then trace the concept of *decimal expansion* in rational **Q** real **R** number systems for building blocks and origins of calculus.

Our coming discussion need some skill in basic algebra and geometry which added as it is supposed that the reader has. However a lot of concepts will explain on the way.

As we said before a function is a set of ordered pairs as points which produced by a relation

between two sets of domain and range in terms of their Cartesian product. Therefore one can express and graph a function based on this fact on x-y plain. Let's start with a concrete and simple example of a *square functions*  $y = x^2$



The plane is  $\mathbf{R} \times \mathbf{R}$ , or  $\mathbf{R}^2$  and the function  $f(x) = x^2$  chose values from the set of  $\mathbf{R}$  on the x-axis as  $x \in \mathbf{R}$  and give some values of  $f(x) \in \mathbf{R}$ . Therefore it is a mapping from  $x$  to  $f(x)$  which occurs at  $(x, f(x))$  as an ordered pair which is a **point**. This make us the graph of the function  $f$  which is the set (subset) defined as:

$$\{(x, f(x)) \mid x \in \mathbf{R}\}.$$

According to this illustrations we can define every function we need such as quadratic, cubic,

rational, exponential etc. In fact they are some algebraic equations along with two sets introduced to them to act as a function. Therefore we deal with intervals of values in the domain set as domain and interval of values in the codomain set as range. Thus one can say that a function is describing the **pattern** (map) of values or variables of a set, event or phenomenon.

We define a function by a table of values, graphs and formula which is called an **analytic expression**. This can be composed of different formulas that chose unique, finite real independent variables from domain space. In analytic representation of a function we have three usual forms: **explicit**, **implicit** and **parametric**.

If the dependent variable  $y$  is expressed in the form  $y = f(x)$  then  $y$  is an *explicit* function of  $x$ . for example,  $y = 5x + 1$  is explicit but  $5x - y + 1 = 0$  is not, though it can be rearranged to be explicit.

If the independent variable is not the only term on one side of an equation and express in the form  $f(x, y) = 0$  i.e the value of  $f$  is dependent on  $x$

and  $y$  values such as in  $3x + 5y - 1 = 0$  then this is an *implicit* form while the equivalent equation  $y = 0.6x + 0.2$  is explicit.

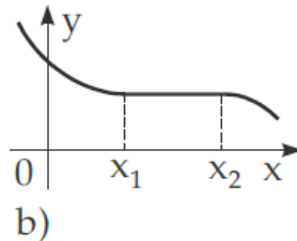
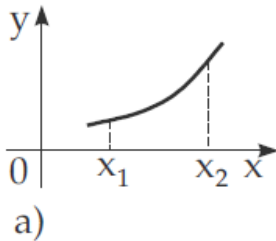
If corresponding values of  $x$  and  $y$  are given as functions of (be dependent on) an auxiliary variable  $t$  which is called a *parameter*, then the functions such as  $\varphi(t)$  and  $\phi(t)$  must have the same domain. For instance;

$$x = t + 2\sin t = \varphi(t), y = t - \cos t = \psi(t)$$

Here  $x = \varphi(t)$  and  $y = \phi(t)$  why? Since the value of *sine* and *cosine* depended on the values of  $t$  on every point.

A function can be *monotonic*, *bounded*, *even/odd*, *periodic* and *inverse*.

A real function  $f$  is monotonically increasing or decreasing on an interval  $I: (f(x_1) \leq f(x_2))$  (a) or  $I: (f(x_1) \geq f(x_2))$  (b) if  $x_1 < x_2$ .



Also  $f$  is strictly increasing or strictly decreasing as:  $I:(f(x_1) < \text{or} > f(x_2))$  when the equality never holds.

In addition to ordered pairs we have triples, quadruples, and so on as our involved dimensions, sets or spaces grow. Therefore we need a more general notation than function i.e. a *sequence* which can be *finite* or *infinite*. A *finite sequence* consists of  $n$  terms  $a_1, a_2, \dots, a_n$ , one corresponding to each of the integers  $1, 2, \dots, n$ , where  $n$ , is some positive integer, representing total number of terms in the sequence while in *infinite sequence* these terms  $a_1, a_2, a_3, \dots$ , are infinite i.e. an endless list of terms. Formally we have a sequence: If  $X_n$  is an ordered set

$$\{1, 2, 3, \dots, n\} = \{x \in \mathbf{N} | 1 \leq x \leq n\}.$$

Now: If  $S$  is a set then an  $n$ -tuple of elements of  $S$  (the sequence) is defined to be a function  $f: X_n \rightarrow S$ ; a mapping from the set  $X_n$  to the set  $S$ . therefore the elements of  $S$  would represent as:  $f(1), f(2), \dots, f(n)$  or  $(f_1, f_2, \dots, f_n)$  so  $n$ -tuple of elements.

A sequence is a given rule by which any term of the list can be uniquely determined then one must

seek for a formula for general term of sequence  $(a_n)$  such as:

$$a_n = n: 1, 2, 3, \dots$$

$$a_n = (-1)^{n+1}n: 1, -2, 3, -4, 5, -6, \dots$$

Therefore we can also have increasing or decreasing monotonic sequence if

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$$

And

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$$

Hold respectively. And if equality never hold then we have strictly increasing or strictly decreasing sequences.

As it is mentioned earlier a *bounded above* if there exist an upper bound (a number) which the values of the function never exceed it. This is also true for a lower bound such that the values of the function are never less than this number. If it is both is hold we say the function is *bounded*. The important point is that: If there exist one upper bound for a function then it has an infinite number of upper bounds i.e. all numbers greater than this one and among them there exist always a smallest

one, called *least upper bound*. This is also true for a lower bounds.

$y = e^x$  is bounded below ( $y > 0$ )

$y = 1 - x^2$  is bounded above ( $y \leq 1$ )

$y = \sin x$  is bounded ( $-1 \leq y \leq +1$ )

Therefore *boundedness* is come from or based on the properties of *order relation* we said before.

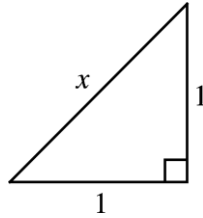
If  $\exists k \leq a_n \forall n$  then  $k$  is the *lower bound* of  $a_n$ .

If  $\exists k \geq a_n \forall n$  then  $k$  is the upper bound of  $a_n$ .

If  $|a_n| < k$  then  $a_n$  is bounded. If such a  $k \nexists$  then  $a_n$  is unbounded.

Based on this discussion and the existence of the set of real numbers  $\mathbb{R}$  we can build a rigorous foundation for existence of convergence and limit for a function and sequence.

However the initial point is whence the length of an object couldn't be measured by a rational number such as in the case of a right triangle below;



To compute  $x$  or hypotenuse we apply Pythagoras theorem then:  $x^2 = 1^2 + 1^2 = 2$  i.e.  $x^2 = 2$  but  $x$  cannot be rational, because there is no rational number  $m/n$  such that  $(m/n)^2 = 2$  where  $m$  and  $n$  are two integers. And also if you continue to solve this equation you get;  $x = \sqrt{2}$  and this number is not rational. Therefore the lengths like,  $\sqrt{2}, \sqrt{3}, \pi, e \dots$  enlarge our number system further to a combination of rational and irrational numbers called real number system **R** which we illustrated earlier.

Therefore **R** is founded based on decimal expansion of rational number. Then in theory, one can represent any real number as a decimal expansion he/she need.

However one can say "this is not true since every point on the line occupied a place or position as a number actually is so there is no real number existed at all and all numbers are rational" but

remember we are in the world of decimal expansions of position (number) and there exist infinite numbers of incomplete decimal and fractions so infinite irrational numbers which is larger than complete decimal and fractions i.e. rational numbers. For instance the position of  $\sqrt{2}$  is among:  $1.4142 < \sqrt{2} < 1.4143$ . Generally this point can be showed as:

$$\begin{aligned} a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots \frac{a_n}{10^n} &\leq x \\ &< a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots \frac{a_n + 1}{10^n}. \end{aligned}$$

Then we have a "succession of intervals" or *sequence*, of positions  $10^{-1}$ ,  $10^{-2}$ ,  $10^{-3}$ , ..., each containing the point  $x$ . This fact can be defied *analytically* with the help of *completeness or least upper bound axiom* of real numbers. Remember "least upper bound" is the smallest number among all upper bound in sequence or function. Thus exact representation of the number  $x$  requires a decimal expansion of  $a_0.a_1a_2a_3..$  that goes on forever or to the first  $n$  decimal places to form an infinite decimal which is called a *real number* **R** or better as:

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots \frac{a_n}{10^n},$$

So every  $a_n$  is a decimal place. For example:

$$\frac{1}{8} = 0.125000 \dots$$

But since two different sets of real numbers can have the same supremum then a real number may have two different decimal expansion. This is called *Archimedes' Condition*: Given an *arbitrarily small* positive real number  $\varepsilon$ , there exists a positive integer  $n$  such that  $\frac{1}{10^n} < \varepsilon$ . For instance our above example can also be represented as:

$$\frac{1}{8} = 0.124000 \dots$$

In fact if two real numbers have the same decimal expansion to  $n$  places, then they differ by *at most*  $1/10^n$  i.e.

$$-\frac{1}{10^n} < x - y < \frac{1}{10^n}$$

If we put  $\frac{1}{10^n} \equiv \varepsilon$  as an accuracy to our decimal expansion then we can change the above expression as:

$$a - \varepsilon < x < a + \varepsilon$$

$$-\varepsilon < x - a < \varepsilon$$

$$|x - a| < \varepsilon$$

Note is that the minus and plus sign does not mean as usual but they mean a *place* before and after the value of  $x$  then  $|x - a| < \varepsilon$  is the distance to  $\varepsilon$  in both sides. In example of

$$\frac{1}{8} = 0.125000 \dots \quad \text{and} \quad \frac{1}{8} = 0.124000 \dots$$

becomes:  $0.125 < \frac{1}{8} < 0.124$ .

Based on these illustrations we can express a real number  $L$  as a **limit** for  $a_n$  instead of being a particular decimal expansion. An infinite sequence of numbers i.e.

$a_1, a_2 \dots a_n \dots = \{a_k\}; k = 1, 2, \dots\}$  has a limit  $L$  if for an infinite increase of the  $(n)$  the index, difference  $a_n - l$  becomes arbitrarily small i.e. the more grows of decimal places which are indexed with  $(n)$ , the more closer and closer to the value of limit  $L$ . formally we say: for an arbitrary small  $\varepsilon > 0 \exists N < n \forall n$  where  $N$  is a natural number. For instance decimal expansions of  $1.4142 \dots \sqrt{2}$  are getting closer and closer

together, until they become *indistinguishable* from each other and from  $\sqrt{2}$  up to the accuracy we need. Generally this is called *convergence* to  $\sqrt{2}$ .

Therefore for *convergence* of  $(a_n)$  to a limit  $l$ , with an accuracy  $\varepsilon$ , there must exist some natural number  $N$  such that the difference between  $a_n$  and  $l$  has size less than  $\varepsilon$  when  $n > N$ . In other words,  $|a_n - l| < \varepsilon$ . formally we define:

A sequence  $(a_n)$  of real numbers tends to or *converges* to a limit  $l$  if, given any  $\varepsilon > 0$ , there is a natural number  $N$  such that:

$$|a_n - l| < \varepsilon \quad \forall n > N$$

Or

$$\lim_{n \rightarrow \infty} a_n = l \text{ or } a_n \rightarrow l \text{ as } n \rightarrow \infty$$

Non-convergent sequences of numbers are called *divergent*.

Based on the arithmetic of decimal such as:

$2/3 + 2/7 = 20/21 = 0.952380$ . we can have arithmetic of sequences;  $a_n, b_n, c_n \dots$  as:

$$(a_n \pm b_n) = (a_n) \pm (b_n)$$

$(a_n/b_n) = (a_n)/(b_n)$  if  $b_n \neq 0$  hold

These operations perform on each pair of terms in corresponding positions between  $a_n$  and  $b_n$ ...

One can do these operations for the convergent sequences as well.

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n,$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \left( \lim_{n \rightarrow \infty} b_n \right)$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

If  $b_n \neq 0$  for every  $n$ , and  $\lim_{n \rightarrow \infty} b_n \neq 0$

All of these can be expanded to a new concept called *series* which generally are summation process of terms in a sequence such that if an infinite sequence of numbers is;

$\{a_k\} = a_k = a_1 + a_2 + a_3 + \cdots a_n + \cdots$  Then  $\sum_{k=1}^n a_k$  is an infinite series which composed all terms of  $a_k$ . If a part of  $a_k$  i.e.  $S_n = a_1 + a_2 + \dots a_n$  is summed;

$$S_1 = a_1, S_2 = a_1 + a_2, \dots S_n = \sum_{k=1}^n a_k$$

Then it is a *finite series* and a *partial sum* as well, which is in fact the sum of the first  $n$  terms. A general, random sequence of numbers can be described as a series and a sum of the terms found as a rule of summation as some sort of relationship between successive terms such as:

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + \dots n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + \dots n^3 = \frac{1}{4}n^2(n+1)^2$$

If the value of the partial sum  $S_n$  tends to a finite *limit*,  $S$ , as  $n$  tends to infinity, then the series is said to *converge* and its sum is given by the limit  $S$ . In fact a series is called *convergent* if the sequence of partial sums  $\{S_n\}$  is convergent. This is represented as:

$$S = \lim_{n \rightarrow \infty} S_n = \sum_{k=1}^{\infty} a_k$$

If this is not hold and the limit does not exist or it is equal to  $\pm\infty$  then we call the series *divergent*.

However some may oscillate such as in periodic series such as Fourier series.

The concept of *limit* is simply about approaching to a point or value but never exceed it. Formally we say: A function  $y = f(x)$  has a limit  $L$  at  $x = a$  as:

$$\lim_{x \rightarrow a} f(x) = L \text{ or } f(x) \rightarrow L \text{ for } x \rightarrow a$$

In fact as  $x$  gets closer and closer to  $a$ ,  $f(x)$  gets closer and closer to  $L$ . therefore  $x$  is approaching to  $a$ , while  $f(x)$  or  $y$  is approaching to  $L$ . More precisely:  $f(x)$  is infinitesimally closing to  $L$  while  $x$  is infinitesimally closing to  $a$ . Thus,  $f(x)$  tends to  $L$  as  $x$  tends to  $a$ :

$$f(x) \rightarrow L, \text{ as } x \rightarrow a,$$

Now if there exist any positive number  $\varepsilon$ , there is a positive number  $\delta$  which depends upon  $\varepsilon$  such that, for all  $x$ , but not  $a$ , lying between  $a - \delta$  and  $a + \delta$ ,  $f(x)$  lies between  $L - \varepsilon$  and  $L + \varepsilon$ . This is called delta-epsilon definition of a limit for a function. More formally we say:

$$f(x) \rightarrow l, \text{ as } x \rightarrow a, \exists \varepsilon \& \exists \delta : \forall x \neq a \\ : a - \delta < x < a + \delta \Rightarrow a - \varepsilon < f(x) < a + \varepsilon$$

Or succinctly:

$\lim_{x \rightarrow a} f(x) = L$  means that given any  $\varepsilon > 0$ ,  
a number  $\delta > 0$  can be found so that  
if  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \varepsilon$ . If

you take a ratio and instead of separate symbols of infinitesimal motion use  $\Delta$  then you get a ratio which is equivalent to the slope formula of a line and a linear equation as  $m = \frac{\Delta y}{\Delta x}$ . Now you can evaluate the slope of a line or a curve instead of say only the difference of two point and take a ratio of it as a slope since you are moving on the curve infinitesimally and this is called *differentiation*.

Based on these definitions we can have properties of limit as:

$$\lim_{x \rightarrow a} c = c \quad \lim_{x \rightarrow a} x = a$$

If  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$ , then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

$$\lim_{x \rightarrow a} [f(x) - g(x)] = L - M$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = LM \quad \lim_{x \rightarrow a} [f(x)]^n = L^n$$

$$\lim_{x \rightarrow a} [f(x)/g(x)] = L/M \quad \text{provided } M \neq 0$$

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

**Provided  $n$  is an odd integer, or  $n$  is an even integer and  $L$  is positive.**

Some example of finding limit:

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 1) = 3.$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{x(\sqrt{1+x} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{x(\sqrt{1+x} + 1)} = \frac{1}{2}. \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{2(\sin 2x)}{2x} = 2 \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} = 2.$$

The exponential function  $\exp(x) = e^x$  tends to infinity more quickly than any high power  $x^n$  and  $n$  is a fixed positive number. This can be shown/prove by using *l'Hospital's rule* which is used to find the limit of a rational functions in *indeterminate forms* as bellow:

$$\lim_{x \rightarrow \infty} \left| \frac{e^x}{x^n} \right| = \infty$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \lim_{x \rightarrow \infty} \frac{e^x}{nx^{n-1}} = \dots = \lim_{x \rightarrow \infty} \frac{e^x}{n!} = \infty$$

The *l'Hospital's rule* use differentiation rules in nominator and denominator to find or prove the limit which comes later.

Along with the *limit* we have *continuity* for a function. Roughly it means the graph of a function have no gap or hole as a *discontinuity* and someone can draw the curve without lifting his/her pencil. But formally a function  $f$  is continuous in an open interval if it is continuous at each point of the interval; and  $f$  is continuous

on the closed interval  $[a, b]$ , where  $a < b$ , if it is continuous in the open interval  $(a, b)$  and if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \text{ and } \lim_{x \rightarrow b^-} f(x) = f(b),$$

The following properties hold:

1 The *sum* and *product* of two continuous function is continues. Such as any polynomial function.

2 The *quotient* of two continuous functions is continuous at any point or in any interval where the denominator is not zero. Such as any rational function.

We can also use continuity to prove important theorems such as bellow

a) If the function  $f(x)$  takes every value between A and B on the interval  $(a,b)$  at least once then it satisfies *Intermediate Value Theorem*. For this  $f(x)$  must be continuous everywhere in the interval.

b) For a continuous strictly monotone one-to-one function  $f(x)$  on an interval there also exists a continuous strictly monotone inverse function  $f^{-1}(x)$ .

c) If a function  $f(x)$  is continuous on a finite closed interval  $[a,b]$  then it is bounded on this interval. This means there exist two numbers  $m$  and  $M$  such that:

$$m \leq f(x) \leq M \text{ for } a \leq x \leq b$$

With these discussions we turn into *differentiation* and *integration*. Simply the first is talking about slopes or infinitesimal (very small) rate of changes in function or series and the second is talking about adding up or sum up these little changes to get the function again so is an approximation or guessing process. These are extended from basics to advanced level in new branches of calculus such as calculus of **probability** and **stochastic (random) process** however the foundations are the same.

As we said *differentiation* is a process of determining how quickly or slowly a function varies with respect to changes of an independent variable. We say  $f(x)$  is a function since the value of  $f$ , the function depends on the value of  $x$ , the variable. Now this function is continuous on an interval and we wish to find the rate of changes in independent variable with respect to changes in

dependent variable. Also in the definition of limit we remember the definition of slope as:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \tan \theta$$

Now we want to know the value of this ratio in every point of an interval so instead we say:  $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$  which can be extended to  $\frac{\partial y}{\partial x}$  or  $\frac{\delta y}{\delta x}$  for multivariable functions. The first is ordinary differentiation with its *ordinary differential equations* and the second is partial differentiation with its *partial differential equations*.

Formally we define derivative of a function as the *instantaneous* rate of change of the function with respect to the variable  $x$  as:

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

As you notice we have a collection of all notations and concepts we said until now. This can also be simpler with the help of *tangent line*. The tangent line to the graph of a function  $f(x)$  at the point  $(a, f(x))$  is the straight line through the point with the

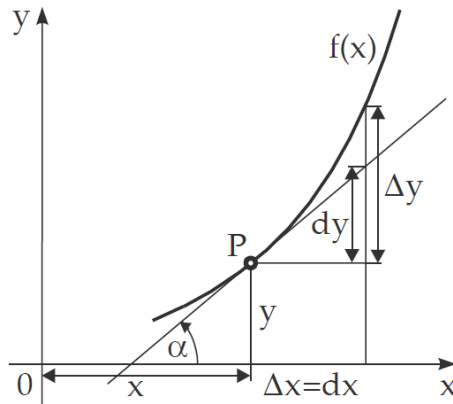
slope  $m$  equal to the derivative of the function at the point  $a$  as:

$$m(a) = f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

This deferential quotient or derivative of a function  $y = f(x)$  at a point  $a, x, x_0, \dots$  is equal to the *limit* of the function when infinitesimal changes ( $\Delta$ ) or  $h$  of independent variable  $x$  tends to zero. Therefore this limit must be existed and finite which means  $f$  must be continuous at the point. Another notations for derivative of a function are as bellow:

$$y', \dot{y}, Dy, \frac{dy}{dx}, f'(x), Df(x), \text{ or } \frac{df(x)}{dx}$$

The value of every  $x$  is equal to the limit of the quotient of the *increment of the function*  $\Delta y$  and the corresponding increment  $\Delta x$  for  $\Delta x \rightarrow 0$  if this limit exists. We can summarize all of these in a graph bellow:



With  $f(x)$  replaced by  $f'(x)$  the second derivative is defined by:

$$f''(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x},$$

This process can be continued if needed as:

$$f^{(n)}(x) \equiv \lim_{\Delta x \rightarrow 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x}.$$

Using definition:

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

One can find the derivative of every function which is differentiable such as  $f(x) = x^2$  with respect to  $x$ .

$$\begin{aligned}
\frac{df(x)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} 2x + \Delta x = 2x
\end{aligned}$$

As  $\Delta x$  tends to zero,  $2x + \Delta x$  tends to  $2x$ , thus:

$$f'(x) = 2x.$$

This process is the same for other functions such as:

$$\frac{d}{dx} (x^n) = nx^{n-1}, \quad \frac{d}{dx} (e^{ax}) = ae^{ax},$$

$$\frac{d}{dx} (\ln ax) = \frac{1}{x}, \quad \frac{d}{dx} (\sin ax) = a \cos ax,$$

Remember:  $f(x)$  is differentiable with respect to  $x$  for the values of  $x$  where the differential quotient has a finite value. This is actually what differentiability mean; "the differential quotient has a finite value" at a particular point on the graph. If we chose an interval from function set to

differentiate, then the domain of the derivative function is a subset of the domain of the original function.

The derivative of a constant function is zero. If it is multiplying by a derivative function it will factor out. There are general basic rules to differentiate in sum, product etc. operations as bellow by separation. The purpose of the separation is to split the whole function into two (or more) parts to have easier differentiation. If  $f(x)$  is written as the product  $u(x)v(x)$ , then

$$\begin{aligned} f(x + \Delta x) - f(x) &= u(x + \Delta x)v(x + \Delta x) \\ &\quad - u(x)v(x) \\ &= u(x + \Delta x)[v(x + \Delta x) - v(x)] \\ &\quad + [u(x + \Delta x) - u(x)]v(x), \end{aligned}$$

Now from definition of derivative we have:

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \left\{ u(x + \Delta x) \left[ \frac{v(x + \Delta x) - v(x)}{\Delta x} \right] \right. \\ \left. + \left[ \frac{u(x + \Delta x) - u(x)}{\Delta x} \right] v(x) \right\}. \end{aligned}$$

Then consequently we have;

$$\frac{df}{dx} = \frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x).$$

This can be stated concisely as

$$f' = (uv)' = uv' + u'v.$$

This is a general result and read as;

*"The derivative of the product of two functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first."*

The product rule can be extended to the product of three or more functions.

$$f(x) = u(x)v(x)w(x)$$

By using our previous general relation;

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}[u(x)v(x)] \\ &= u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x).\end{aligned}$$

For the first time we get;

$$\frac{df}{dx} = u \frac{d}{dx} (vw) + \frac{du}{dx} vw.$$

Then for the second time we gain;

$$\frac{d}{dx}(uvw) = uv \frac{dw}{dx} + u \frac{dv}{dx} w + \frac{du}{dx} vw$$

And finally as before we have;

$$(uvw)' = uvw' + uv'w + u'vw.$$

This can be extended to any number  $n$  of factors using *Leibnitz's theorem* which gives the corresponding results for the higher derivatives of products.

In the case of differentiating a function of a function i.e. composition of functions we use the *chain rule*. Here if  $\Delta f$ ,  $\Delta u$  and  $\Delta x$  are small finite quantities then:

$$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta u} \frac{\Delta u}{\Delta x}.$$

Since these quantities are infinitesimally small we get;

$$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}.$$

For example in;

$$f(x) = (4 + t^2)^3 = [u(x)]^3 = u^3$$

Where  $u(x) = 4 + t^2$ .

Then applying the last result above we have:

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du} \frac{du}{dx} = 3u^2 \frac{d}{dx} (4 + t^2) = 3u^2 \times 2t \\ &= 6x(4 + t^2)^2\end{aligned}$$

In the case of quotient expressions we apply *quotient rule*. This rule can be gained using previous relation again;

$$\frac{df}{dx} = \frac{d}{dx} [u(x)v(x)] = u(x) \frac{dv(x)}{dx} + \frac{du(x)}{dx} v(x).$$

For the derivative of a product to a function

$$f(x) = u(x)[v(x)]^{-1}$$

We obtain the derivative of the quotient of two factors.

$$f' = \left(\frac{u}{v}\right)' = u \left(\frac{1}{v}\right)' + u' \left(\frac{1}{v}\right) = u \left(-\frac{v'}{v^2}\right) + \frac{u'}{v},$$

Which is equivalent to a more handy relation as:

$$f' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

For evaluating  $v(x)^{-1}$  we apply the chain rule as;

$$\frac{df}{dx} = -v^{-2} \frac{dv}{dx} = -\frac{1}{v^2} \frac{dv}{dx}.$$

In words the "*derivative of a quotient is equal to the bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared.*"

For example:

To differentiate  $f(x) = \sin x/x$  with respect to  $x$  we use;

$$f' = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

With  $u(x) = \sin x$  and  $v(x) = x$  knowing that  $u'(x) = \cos x$  and  $v'(x) = 1$  we get;

$$f'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

In the case of having implicit function we differentiate term by term and this is called

*implicit differentiation.* For example in differentiating of an implicit equation below with respect to  $x$ :

$$x^3 - 3xy + y^3 = 2$$

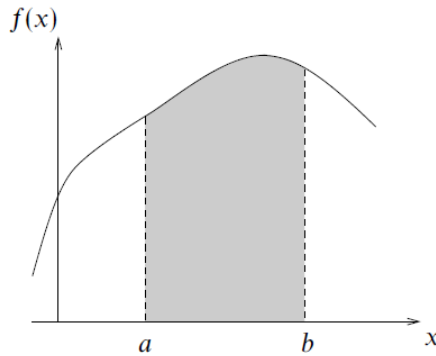
We use this method as:

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(3xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(2)$$

$$3x^2 - \left(3x \frac{dy}{dx} + 3y\right) + 3y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

*Integration* is simply talking about area under a curve, volumes, and surface for sphere, space and other objects. However this is a strong tool in analysis of different theorems and phenomena since it gives us back the original function or something near to it. This is done by adding up those little changes of differentiation process originate from its limit process. While differentiation calculates the derivative function  $f'(x)$  of a given function  $f(x)$  integration determines a function whose derivative  $f'(x)$  is previously given. One can show integration in a simple way as:



$$I = \int_a^b f(x) dx.$$

*Integration* as an area under a curve is known as the *definite integral* of  $f(x)$  since it's integrating (summing up all little changes, slopes or ratios) between the lower limit  $x = a$  and the upper limit  $x = b$ . The function  $f(x)$  is called the *integrand*.

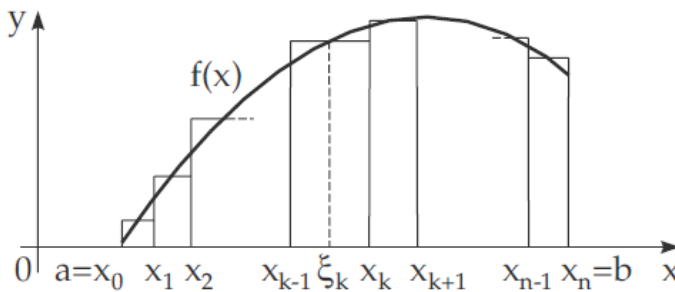
Integrate  $f'$  means 'find an antiderivative of  $f$  which is also called an *indefinite integral* of  $f$  represented by:

$$\int f(x) dx$$

While differentiation more deal with known functions, integration goes beyond because of its

reverse back process to original functions. For instance from integrating particular rational functions you get logarithmic and periodic functions such as trigonometric functions.

Greeks invented a way to exhaust the given region to find its area. They divided the region to many sides and then summed up those sides to a better guess. This is used by Archimedes (287-212 BC.) to find exact formulas for the area of a circle and a few other special figures. Gradually the method of exhaustion was transformed into the subject now called integral calculus, by the help of the decimal expansion of real numbers.



When Integration is used as the inverse operation of differentiation, it determines a function whose derivative is previously given. This process does not have a unique result, so we get the notion of an *indefinite integral*.

The relation between definite and indefinite integrals is the *fundamental theorem of calculus* which says:

If  $f$  is continuous on  $[a, b]$  and  $f$  is a function such that  $f'(x) = f(x)$  for all  $x$  in  $[a, b]$ , then:

$$\int_a^b f(x)dx = f(b) - f(a)$$

This establishes that integration is the reverse process to differentiation. It also can be say as: If the integrand  $f(x)$  is continuous on the interval  $[a, b]$  and  $F(x)$  is a primitive function, then:

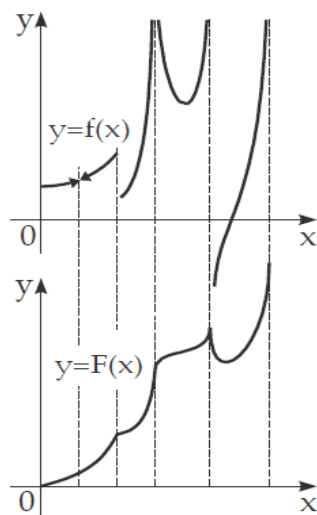
$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$$

Thus, the calculation of a *definite integral* is reduced to the calculation of the corresponding *indefinite integral*, to the determination of the antiderivative:

$$F(x) = \int f(x)dx + C$$

If a function is continuous, definitely it has a primitive function. If there are some discontinuities, then we decompose the interval

into subintervals in which the original function is continuous. The figure below illustrate this:



Some direct properties of definite integrals that are as follows:

$$\int_a^b 0 \, dx = 0, \quad \int_a^a f(x) \, dx = 0,$$

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx,$$

$$\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Indefinite integrals when they are inverse process of differentiation as:

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

The integral of an integrand of arbitrary elementary functions is not usually an elementary function. In such a case we use rules of integration. Some basic rules are:

$$\int \alpha f(x) dx = \alpha \int f(x) dx.$$

The constant can be factor out. This is *Constant multiple rule*.

$$\int (u + v - w) dx =$$

$$\int u dx + \int v dx - \int w dx.$$

*Sum*

*rule* by integrating separate terms (integration by parts). This is also true for difference so then:

$$\int [u(x) \pm v(x)] dx = \int u(x) dx \pm \int v(x) dx$$

And partial integration as:

$$\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx$$

Indefinite integral of  $f(x)$  is represented as:

$$\int f(x) dx = F(x) + c$$

With this we can have:

$$\int a dx = ax + c, \quad \int ax^n dx = \frac{ax^{n+1}}{n+1} + c,$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + c, \quad \int \frac{a}{x} dx = a \ln x + c,$$

$$\int a \cos bx dx = \frac{a \sin bx}{b} + c,$$

$$\int a \sin bx \, dx = \frac{-a \cos bx}{b} + c,$$

Let's do some samples:

$$\begin{aligned} \int (x+3)^2(x^2+1) \, dx &= \\ \int (x^4+6x^3+10x^2+6x+9) \, dx &= \\ = \frac{x^5}{5} + \frac{3}{2}x^4 + \frac{10}{3}x^3 + 3x^2 + 9x + C. \end{aligned}$$

$$\int \sin 2x \cos x \, dx = \int \frac{1}{2}(\sin 3x + \sin x) \, dx.$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C.$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C.$$

$$\int \frac{2x+3}{x^2+3x-5} \, dx = \ln|x^2+3x-5| + C.$$

$$\int \frac{2x+3}{(x^2+3x-5)^3} \, dx = \frac{1}{(-2)(x^2+3x-5)^2} + C.$$

In our earlier discussions we considered set theory and showed that how logic, functions and calculus are related to it. Now we go on with **probability, combinatorics**.

According to *oxford dictionary of mathematics* "the probability of an event  $A$ , denoted by  $Pr(A)$ , is a measure of the possibility of the event occurring as the result of an experiment. For any event  $A$ ,

$$0 \leq Pr(A) \leq 1.$$

If  $A$  never occurs, then  $Pr(A) = 0$ ; if  $A$  always occurs, then  $Pr(A) = 1$ . If an experiment could be repeated  $n$  times and the event  $A$  occurs  $m$  times, then the limit of  $m/n$  as  $n \rightarrow \infty$  is equal to  $Pr(A)$ .

If the sample space  $S$  is finite and the possible outcomes are all equally likely, then the probability of the event  $A$  is equal to  $n(A)/n(S)$ , where  $n(A)$  and  $n(S)$  denote the number of elements in  $A$  and  $S$ . The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.

The probability that a discrete random variable  $X$  takes the value  $x_i$  is denoted by  $\Pr(X = x_i)$ . The probability that a continuous random variable  $X$  takes a value less than or equal to  $x$  is denoted by  $\Pr(X \leq x)$ . This notation may be extended in a natural way".

The counting aspect of combinatorics is very useful in finite probability theory, where the probability of an event is found by counting the number of favorable cases and comparing that number to the total number of cases. For example, the probability of throwing a total of 12 with two dice is  $1/36$ , since there is one favorable case (6 with each die) out of 36 possibilities. On the other hand, the probability of throwing a total of 8 is  $5/36$  because there are five favorable cases:  $2+6$ ,  $3+5$ ,  $4+4$ ,  $5+3$ ,  $6+2$ , where the first number in each sum is the number on the first die, and the second number is the number on the second die.

A probability space is a finite **measure** space with associated probability measure that assigns unit measure to the complete space. And a **measure (S) is defined as** a function  $\mu$  defined on a set  $S$  which assigns a non-negative value from the

extended real numbers i.e. the set  $\mathbb{R}$  in addition to infinities, to each subset with the properties that

$$\begin{aligned}\mu(\phi) &= 0, \mu(A \cup B) \\ &= \mu(A) + \mu(B) \text{ if } A \cap B = \phi.\end{aligned}$$

The standard notion of probability is therefore a special case, where the function takes only values in the interval  $[0, 1]$ . A function with the above properties but taking negative values also is called a signed measure. Measures and measure theory is a fundamental part in every scientific study.

We also can define **measurable function** as a function  $f$  with a domain  $D$  mapping into the set of real numbers is measurable if for every real number  $k$  the set  $\{x \in D: f(x) > k\}$  is measurable. While a **set is measurable** If  $\mu$  is a measure defined on a  $\sigma$ -algebra,  $X$ , of a set  $S$ , then the elements of  $X$  are called  $\mu$ -measurable sets or measurable sets.

With this formal introduction mainly taken from *oxford dictionary of mathematics* we go on with an informal then formal point of view in the case of probability theory.

When we say that under given conditions the success of or happening of a process is 92% or 0.92 we mean that from the whole and complete i.e. 100 % this percent of 92 is hold on the **average**. While the probability of reject or failure is 0.016 or 1.6%. However, this is true and hold when the whole number of events is sufficiently large. Thus "we have to deal with probabilities of only a finite similar number of events or an infinite number of random events in an *experiment* or a *trial*."

In the case of tossing a coin the results are unpredictable since they are **random events** with the probability interval of;

$$0 \leq P(A) \leq 1.$$

And this simple expression is the most important property of probability. Thus, "the probability of an impossible event is zero, the probability of a .sure event is one, and the probability  $P(A)$  of a random event  $A$  is a certain number lying between zero and one".

In simple cases, the probability of any event  $A$  in this experiment or trial with sufficiently large similar number of events is "*the ratio of the*

*number of chances favorable for event A to the total number of chances":*

$$P(A) = \frac{m_A}{n}$$

Where  $n$  is the total number of chances and  $m_A$  is the number of chances favorable for event A.

While in the case of frequency instead of chance;

*"The frequency of an event in a series of  $N$  repetitions is the **ratio** of the number of repetitions, in which this event took place, to the total number of repetitions".*

$$P^*(A) = \frac{M_A}{N},$$

Where  $N$  is the total number of repetitions of the experiment and  $M_A$  is the number of repetitions in which event A occurs (in other words, the number of occurrences of event A).

Note: the word **ratio** pave the way for interring calculus to this scene.

There is a relation between the frequency of an event and its probability. "The most probable events occur more frequently than events with a low probability. Nevertheless, the concepts of

frequency and probability are not identical. The greater is the number of repetitions of the experiment the more marked is the correlation between the frequency and the probability of the event. If the number of repetitions is small, the "frequency of the event is to a considerable extent a random quantity which can essentially differ from the probability".

However, "with the increasing number of repetitions the frequency of the event gradually loses its random nature. Random variations in the conditions of each trial cancel each other out when the number of trials is great; the frequency tends to stabilize and gradually approaches, with small deviations, a certain constant. It is natural to suppose that this constant is exactly the *probability* of the event". This concept is equivalent with that of large number of similar events related to probability. In fact **"Stabilization of frequency** with a great number of repetitions of an experiment is one of the most typical properties observed in mass random phenomena". Thus *as the number of trials increases, the frequency of the event approaches its probability.*

The important point is that if "we can make a sufficient number of similar trials and the frequency of an event is stable, then we can use the frequency of the event in the given **series** of trials as an approximation of the probability of this event". In fact this is a *limiting* relationships between these two. The more deviations of frequency from probability the more practically impossible probability in contrast the lesser deviation of frequency from probability the more practically possible probability to happen for an event.

If an event  $A$ , appears with probability  $p$ , in  $N$  trials then, "with the probability of 0.95, the value of frequency  $P''(A)$  of event  $A$  lies in the confidence interval of;

$$p \pm 2 \sqrt{\frac{p(1 - p)}{N}}.$$

For the frequency of the event, corresponding to the confidence level of **0.95**. "This means that our prediction that the value of frequency will be in the limits of this interval is true in almost all cases, to be exact, in 95% or 0.95 of cases". Thus

the value or amount of error is 5% or 0.05. If this is too high and we need more confidence then;

$$p \pm 3 \sqrt{\frac{p(1-p)}{N}}.$$

This confidence interval corresponding to a very high confidence level of **0.997**. However this can be continued to the 100% or 1. Thus according to Bernoulli theorem of the **law of large numbers** "if the number of independent trials is sufficiently large, then with a practical confidence the frequency will be as close to the probability as desired".

In brief one can say "the frequency of an event tends to its probability ". If one is going to find approximate value of the probability of the event by its frequency and estimate the error of this approximation, then  $P \approx P^*$ .

Now we go on with some basic algebraic rules of probability which is based on algebraic rules of set theory.

The probability that one of two **mutually exclusive events** (it does not matter which of

them) occurs is equal to the sum of the probabilities of these events;

$$P(A \text{ or } B) = P(A) + P(B).$$

In the case of several mutually exclusive events;

$$\begin{aligned} P(A_1 \text{ or } A_2 \text{ or } \dots \text{ or } A_n) \\ = P(A_1) + P(A_2) + \dots + P(A_n). \end{aligned}$$

Thus if  $A$  is an event and  $\bar{A}$  is the opposite event

$$P(A) + P(\bar{A}) = 1$$

This means that the sum of the probability of occur and not occur is equal to one or "the sum of the probabilities of opposite events is equal to one".

The probability of the **combination** of two events that is, of the simultaneous occurrence of both of them is equal to the probability of one of them multiplied by the probability of the other provided that the first event has occurred. This is Probability multiplication rule;

$$P(A \text{ and } B) = P(A) \cdot P(B/A),$$

Where  $P(B/A)$  is the so-called *conditional probability* of an event  $B$  calculated for the condition (on the assumption) that event  $A$  has occurred. This can also as;

$$P(A \text{ and } B) = P(B) \cdot P(A/B).$$

If this dependence does not exist between  $A$  and  $B$  or  $B$  and  $A$  then they are called *independent* i.e.

$$P(A/B) = P(A).$$

For a sequence of independent events the multiplication rule is;

$$\begin{aligned} P(A_1 \text{ and } A_2 \text{ and } \dots \text{ and } A_n) \\ = P(A_1) P(A_2) \dots P(A_n), \end{aligned}$$

Which means that "the probability of the combination of independent events is equal to the product of their probabilities".

**Random variables** are similar to random events i.e. they are existed or distributed by chance. However the difference is that they random variable as it is clear, are quantities which distributed or existed randomly or by chance or even in more technical term **stochastically**. In

the case of an experiment they result various values "which are unknown beforehand".

**Discrete random variables** are possible separated values "from one another by some interval" on the number axis. While continuous random variables are continuous values "fill a certain interval on the number axis". They can form and form in any set of numbers i.e.  $\mathbf{N}$ ,  $\mathbf{Q}$ ,  $\mathbf{C}$  etc.

Thus "the boundaries of the interval are sometimes sharp and determined and sometimes blurred and undetermined".

Suppose we have a set,  $X$  of random values which are not equally likely or probable. Then we can define a *distribution function* for this set of random variables,  $X$ . this distribution function "describes the distribution of the probabilities among the values of this set of  $X$ . In fact it is a model or approximation of the behavior of the corresponding probable distributed random variables on the set of all possible values.

In the case of discrete random variables the distribution can be represented with the help series. This series is called *distribution series* and

use for the set of possible values  $\{x_i\}$  as  $x_1, x_2, \dots, x_n$  and also for the set probable values  $\{p_i\}$  as  $p_1, p_2, \dots, p_n$  corresponding to each other. Thus;

$$P_i = P(X = x_i) \quad (i = 1, \dots, n).$$

And

$$\sum_{i=1}^n p_i = 1$$

Therefore this is **unity** distributed as random variables over the space of the set of possible values!! We also must note that the values of these sets are *uncountable* and if "we try to assign a certain probability to each individual value of a continuous random variables, we would find that this probability equals zero!"

For instance in a set of pebbles you going to find an exactly a pebble of 20 gram and the probability of finding a pebble with this character is zero. This is due to the fact that; any event with zero probability is NOT impossible! However we mentioned that "the probability of an impossible event is zero". Now we face with "possible events

whose probabilities are zero". This is similar with zero area of a point in any nonzero area or space. Thus we can conclude that any space or area are consisted of points with zero area.

We mentioned before that the "probability of getting at every individual point for a continuous random variable is exactly zero". And now we then ask "how can we speak of the *probability distribution* for a continuous random variable?"

The answer is; we use **probability density**. In physics the density of a substance is a ratio of mass per unit volume. In the case of inhomogeneous substance we consider local density.

In probability theory the local density is, "the probability at point  $x$  per unit length" of the interval. In fact;

Probability density of a continuous random variable  $X$  is the *limit* of the ratio of the probability of getting random variable  $X$  into a *small interval* which tend to zero in the *vicinity* or *neighborhood* of point  $x$  to the length of this interval. In the case of considering the behavior of the whole of this distribution based on

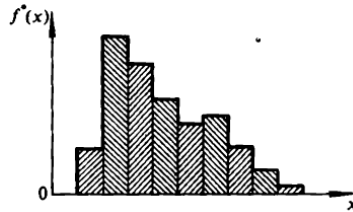
probability density we must find probability density function or PDF.

When we consider the *distribution of probabilities* in an interval of random variables  $X$ , divide the whole range of values of random variable  $X$  into sub-intervals or categories. Then Calculate the number of random variables  $X$  which are existed in "each category and divide by the total number" of say, experiments we made.

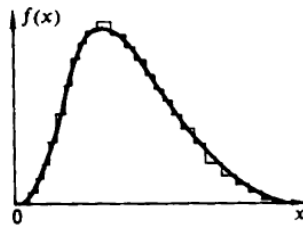
With theses we get "frequency of the category" which the sum of all them should be equal to one. Then by dividing the frequency value by the length of the interval -that can be different- we can calculate the *frequency density* for each interval.

This process is similar to the case of exhaustive method of integration if you remember. Thus plotting our sufficiently large data set we could have a graph such a histogram with bounded area equal to one or a curve with the under-area equal to one. This curve can be resulted from a histogram with increasing shorter intervals i.e. becoming smoother and smoother. Which in fact is natural since we are doing an averaging or

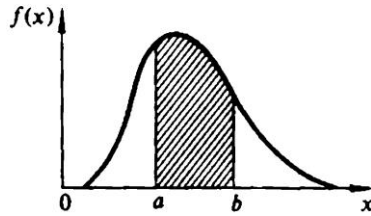
guessing process to approximate probability density value. This curve is called the **distribution curve**. Thus we get the graph below;



Using the process mentioned above this could transform to;



The probability density is located along the y-axis and not the frequency density. "The probability that the set of random variables  $X$  lies within the interval  $(a, b)$  equals the area resting, on this interval" as;



If we denote the probability density function by  $f(x)$ , the probability that the set of random variable  $X$  fits into interval  $(a, b)$  will be expressed by the definite integral:

$$p(a, b) = \int_a^b f(x) dx$$

Therefore having a histogram of available sufficiently large set of data "we can then level it out using some smooth curve" with a bounded unit area. There is a great number of such (fitting) curves in probability theory which can fit with our different set of data to give a good approximation of PDF values which are calculated and recorded in respecting tables or they have easy integral calculations. "In other cases the conditions of the appearance of a random variable suggest a certain type of distribution resulting from theoretical considerations".

The word combination is equivalent with *selection* and **combinatorics** is equivalent with *combinatorial analysis*. Combinatorics is concerned with "counting strategies to calculate the ways in which objects can be arranged to satisfy given conditions".

We start with Khayyam (pascal) triangle which is the binomial expansion of;

$$(a + b)^n$$

As

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

"The numbers in the  $(n + 1)$ st row of the triangle are called *binomial coefficients*" BC i.e.  $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$  where each BC  $\binom{n}{k}$  in the row  $n+1$  is the sum of the two above it and means  $n$  choose  $k$ . thus the combinatorial representation of binomial expansion is;

$$\binom{n}{k} = \frac{n(n-1)(n-2) \dots (n-k+1)}{k!}$$

Where "a sequence of  $k$  choices from a set of  $n$  items" and;

$$k! = k(k-1)(k-2) \dots 3 \cdot 2 \cdot 1.$$

There are  $n(n-1)(n-2) \dots (n-k+1)$  sequences of choices.

Combining this with binomial expansion we can get the famous binomial theorem as;

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{3 \cdot 2}a^{n-3}b^3 + \dots + nab^{n-1} + b^n.$$

When  $a = 1$  and  $b = x$  then;

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \\ \frac{n(n-1)(n-2)}{3 \cdot 2}x^3 + \dots + nx^{n-1} + x^n.$$

Where  $(1+x)^n$  is compact form of the whole sequence "as the coefficients of powers of  $x$ ". This is a generating function for the sequence of binomial coefficients. Thus combinatorics is also

connected with calculus similar to probability theory since it is generally based on finite and infinite sequences and series. However *enumerative combinatorics* deals with listing and counting the elements in finite sets.

Combinatorics in fact is more basic and fundamental than arithmetic since it considers the number of ways, selections, and options of a phenomenon. Thus basically we are dealing with "probability of ways" of *arrangements* and selections. For instance;

There are 30 students in a classroom. What is the probability that all of them have different birthdays?

People born on each 365 day of the year with equal *likelihood*. For determining the number of possible ways of birthdays D for 30 people N, we have  $D/N$ .

Therefore in this topic basically we are dealing with counting and listing i.e. determining the number of elements and listing them in a finite set. This is very useful in cases such as; time analysis of computer algorithms, in statistical mechanics and structural chemistry.

The number of arrangement in a set of  $n$  objects in a given order is called a *permutation* of the object. Formally;

" arrangement of any  $r \leq n$  of these objects in a given order is called an " $r$ -permutation" or "a permutation of the  $n$  objects taken  $r$  at a time." i.e.

$$P(n, r) \quad (\text{other texts may use } {}_n P_r, P_{n,r}, \text{ or } (n)_r).$$

With

$$r \text{ factors in } n(n-1)(n-2) \cdots (n-r+1).$$

Is defined as;

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

For a set of some alike objects; when "the number of permutations of  $n$  objects of which  $n_1$  are alike,  $n_2$  are alike,  $\dots$ ,  $n_r$  are alike i.e.

$$P(n; n_1, n_2, \dots, n_r)$$

We have repetitive Permutations as

$$P(n; n_1, n_2, \dots, n_r) = \frac{n!}{n_1! n_2! \dots n_r!}$$

When order does not matter and we just seeking for the number of combinations of elements or object in a set.

"Let  $S$  be a set with  $n$  elements then a *combination* of these  $n$  elements taken  $r$  at a time is any selection of  $r$  of the elements and is called an *r-combination*". Thus it is a subset of  $S$  with  $r$  elements. Therefore;

$$C(n, r) \quad (\text{other texts may use } {}_nC_r, C_{n,r}, \text{ or } C_r^n).$$

Any combination of  $n$  objects taken  $r$  at a time determines  $r!$  Permutations of the objects in the combination; i.e.

$$P(n, r) = r! C(n, r)$$

Thus

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Consequently

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = C(n, r)$$

In fact we are playing the game of selections, arrangements an assignments. For instance

suppose we have  $n$  pigeons and  $m$  pigeonholes and  $n$  assigned to  $m$  where  $m < n$  "then at least one pigeonhole, contains two or more pigeons". This simple but useful idea is called, **Pigeonhole Principle**. Its extended and general form say;

If  $n$  pigeons are assigned to  $m$  pigeonholes, then one of the pigeonholes must contain at least  $\lceil \frac{n-1}{m} \rceil + 1$  pigeons. This principle is the basis for lots of theorems and results in combinatorics.

This is the end of these part. However we need some more topics in the physics part which we bring them on the need there.

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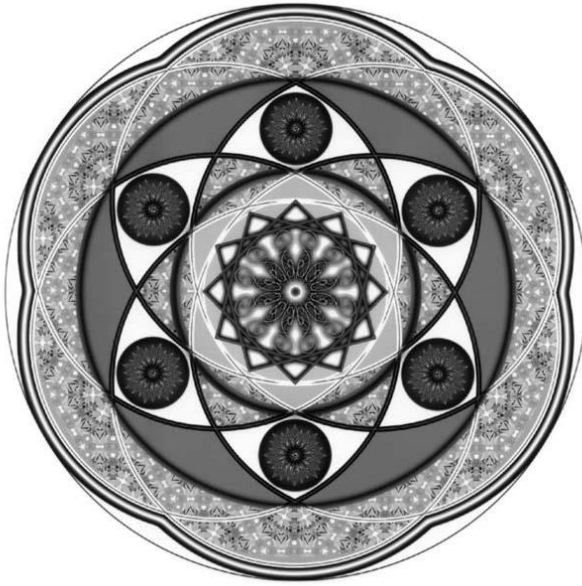
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## Math riddles



Imagination is more important than knowledge.

*Albert Einstein*

Mathematical puzzles and games fulfill the need for diversion, the desire to achieve mastery over challenging subject matter or simply to test our intellectual capacities. In fact it is a mathematical amusement to both the amateur and the professional mathematician. From antiquity to the present this diversions helped to motivate creative incitement. Thus we will do some

puzzles or problems in this way to see how it works.

**1** Diophantus boyhood lasted  $\frac{1}{6}$  of his life; he married after  $\frac{1}{7}$  more; his beard grow after  $\frac{1}{12}$  more, and his son was born 5 years later; the son lived to half his father's age, and the father died 4 years after the son. What was Diophantus age?

### **Solution**

If  $x$ ; represents the number of years he lived, we have the equation:

$$\frac{x}{6} + \frac{x}{7} + \frac{x}{12} + 5 + \frac{x}{2} + 4 = x, \quad \mathbf{x = 84.}$$

Thus, Diophantus married at the age of 33 and had a son who died at the age of 42, four years before Diophantus himself died at the age of 84.

**2** Augustus de Morgan was born in 1806 and was  $x$  years old in the year  $x^2$ . What was his age?

**Solution**  $x^2 = 1806 + x \quad x = 43$

**3** In 4 weeks, 12 oxen consume  $3\frac{1}{3}$  acres of pasture land; in 9 weeks, 21 oxen consume 10 acres of pasture land. Accounting for the uniform growth rate of grass, how many oxen will it

require to consume 24 acres in a period of 18 weeks? (From Newton's book, "Universal Arithmetic".)

### Solution

We represent every quantity in the problem by a letter as bellow:  $\alpha$  - the quantity of grass per acre when the pasture is put into use;  $\beta$  - the quantity of grass eaten by one ox in one week;  $\gamma$  - the quantity of grass that grows in one acre in one week;  $a_1, a_2, a$  - the number of oxen;  $m_1, m_2, m$  - the number of acres;  $t_1, t_2, t$  - the numbers of weeks in the three cases considered, respectively.

Now we can form a system of three equations,

$$m_1(\alpha + t_1\gamma) = a_1t_1\beta,$$

$$m_2(\alpha + t_2\gamma) = a_2t_2\beta,$$

$$m(\alpha + t\gamma) = at\beta,$$

Where  $\alpha, \alpha/\beta, \gamma/\beta$  are unknowns. By solving above system of equation we get:

$$a = \frac{m[m_1a_2t_2(t - t_1) - m_2a_1t_1(t - t_2)]}{m_1m_2t(t_2 - t_1)}.$$

Substituting the numerical data we find **a**  
**= 36**. Therefore, 36 oxen will consume 24 acres  
 in 18 weeks.

**4** Arrows, in the form of thin cylinders with circular cross- section, can be packed in hexagonal bundles. If there exist eighteen circumferential arrows, determine the total number of the arrows to be found (in the bundle) within the quiver.

### **Solution**

Here we have a series to sum as:

$$1 + 1 \cdot 6 + 2 \cdot 6 + 3 \cdot 6 + \cdots + k \cdot 6 + \cdots$$

If 18 arrows are visible in the package, the adding stops when member 18 of the series appears. Therefor the total arrows is:

$$1 + 6 + 12 + 18 = 37$$

**5** Divide a given square number into two squaws.(from Diophantus' Arithmetica)

### **Solution**

Diophantus expressed this in the form of a quadratic polynomial which must be a square as bellow:

If say  $b$  is a given rational number and

$x^2 + y^2 = b^2$  where  $x$  and  $y$  are rational solutions as. To show this Diophantus introduced the substitution  $y = ax - b$ , where is an arbitrary rational number. Thus:

$$b^2 - x^2 = a^2 x^2 - 2abx + b^2,$$

which reduces to:

$$2abx = (a^2 + 1)x^2.$$

Therefore:

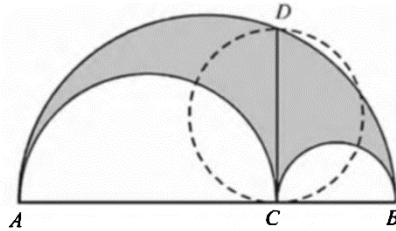
$$x = \frac{2ab}{a^2 + 1}.$$

Let's test our gained formula: if  $b=4$  then  $a = 2$  thus  $x = 16/5, y = 12/5$ . This satisfies the given equation as:

$$\left(\frac{16}{5}\right)^2 + \left(\frac{12}{5}\right)^2 = \frac{400}{25} = 16.$$

**6** The "shoemaker's knife" or arbelos is the region bounded by the three semicircles that touch each other. The task is to find the area which lies inside

the largest semicircle and outside the two smallest (shaded portion in the Figure).



### Solution

Archimedes demonstrated in Proposition 4 of his book of Lemmas that if  $CD$  is perpendicular to  $AB$ , then the area of the circle with diameter  $CD$  is equal to the area of the arbelos. To prove this we say; the triangle  $ABD$  is right-angled since the angle at  $D$  is  $90^\circ$  (as a peripheral angle which corresponds to the diameter  $AB$ ). Then if  $|\cdot|$  denotes the length of a segment we have:

$$(|AC| + |CB|)^2 = |AD|^2 + |DB|^2 =$$

$$|AC|^2 + |DC|^2 + |CB|^2 + |DC|^2,$$

From these we get a well-known result related to the geometrical mean as:

$$|DC|^2 = |AC| \cdot |CB|,$$

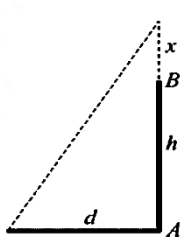
Let  $P(\smile)$  and  $P(\bigcirc)$  be the areas of the arbelos and the circle with diameter  $CD$ . Based on above we get:

$$\begin{aligned} P(\smile) &= \frac{\pi}{8}|AB|^2 - \frac{\pi}{8}|AC|^2 - \frac{\pi}{8}|CB|^2 \\ &= \frac{\pi}{8}(|AC| + |CB|)^2 - |AC|^2 - |CB|^2 \\ &= \frac{\pi}{4}|AC| \cdot |CB| = \frac{\pi}{4}|CD|^2 = P(\bigcirc). \end{aligned}$$

7 On a cliff of height  $h$ , lived two ascetics. One day one of them descended the cliff and walked to the village inn which was distance  $d$  from the base of the cliff. The other, being a wizard, first flew up height  $x$  and then flew in a straight line to the village inn. If they both traversed the same distance, what is  $x$ ?

### Solution

Let  $h = |AB|$  where  $h$  is the height of a cliff. If this represent as a figure bellow then, one can derive the equation as:



$$h + d = x + \sqrt{(h + x)^2 + d^2}$$

Or

$$h + d - x = \sqrt{(h + x)^2 + d^2}.$$

Which after squaring and arranging we get:

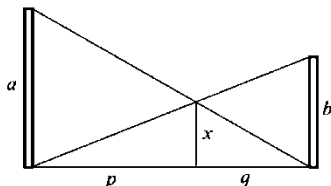
$$2hd - 2dx - 2hx = 2hx$$

And then we get:

$$x = \frac{hd}{2h + d}.$$

**8** Two vertical pillars are of a known height, say  $a$  and  $b$ . Two strings are tied, each of them from the top of one pillar to the bottom of the other, as shown in Figure below from the point where the two strings meet, another string is suspended vertically till it touches the ground. The distance

between the pillars is not known. It is required to determine the height of this suspended string.



### Solution

Let  $p$  and  $q$  be the lengths of distances obtained after dividing the horizontal distance between pillars by the suspended string. Let  $x$  be the unknown height of suspended string. From Figure we have:

$$\frac{p}{x} = \frac{p+q}{b}, \quad \frac{q}{x} = \frac{p+q}{a}.$$

Hence

$$\frac{1}{x} = \frac{1+q/p}{b}, \quad \frac{1}{x} = \frac{1+p/q}{a}$$

Or

$$\frac{b}{x} - 1 = \frac{q}{p}, \quad \frac{a}{x} - 1 = \frac{p}{q}.$$

After multiplication of these two relations we get:

$$\left(\frac{b}{x} - 1\right)\left(\frac{a}{x} - 1\right) = 1,$$

This can be reduced to

$$\frac{ab}{x^2} - \frac{a+b}{x} = 0.$$

Hence we find the height as:

$$x = \frac{ab}{a+b}.$$

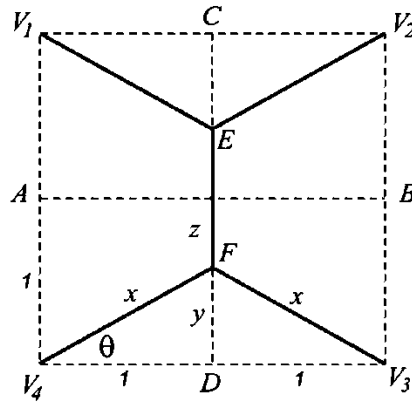
Thus the height  $x$  does not depend on the distance between the pillars. Then the value of  $x$  is easy to find by  $\frac{a}{p} - \frac{b}{a}$ . This means that the horizontal distance between the pillars is divided by the point at which the suspended string touches the ground in the ratio of their heights.

**9** Four villages, each being a vertex of a square, should be connected by a road network so that the total length of the road system is minimal.

### **Solution**

Let  $V_1$ ,  $V_2$ ,  $V_3$ , and  $V_4$  be the vertices of the square of side length equal to 2 units. AB and CD

are perpendicular bisectors like figure bellow. A road system is composed of the solid lines  $EV_1$ ,  $FV_2$ ,  $EF$ ,  $FV_3$ , and  $FV_4$ , with the unknown angle  $\theta$  which should be determined so that this road system has minimal length. Using the triangle inequality it is easy to prove that the symmetric network X shown in Figure has smaller total length than any non- symmetric network X.



By this Figure, the length of the desired road network is:

$$S = 2(2x + z)$$

Since

$$y = \tan \theta, \quad x = \frac{1}{\cos \theta},$$

$$z = 1 - y = 1 - \tan \theta, \quad \theta \in [0, \pi/4],$$

We get:

$$S(\theta) = 2\left(1 - \tan \theta + \frac{2}{\cos \theta}\right) = 2 + 2f(\theta),$$

Then we say:

$$f(\theta) = \frac{2}{\cos \theta} - \tan \theta.$$

Since

$$f(\theta) = (2 - \sin \theta) / \cos \theta > 0 \text{ for } \theta \in [0, \pi/4],$$

to find the minimum of  $S(\theta)$  it is sufficient to determine the minimum of the function  $f(\theta)$  on the interval  $[0, \pi/4]$ . First, suppose that the inequality below is true.

$$f(\theta) = \frac{2}{\cos \theta} - \tan \theta \geq \sqrt{3}, \quad \theta \in [0, \pi/4].$$

Since  $\cos \theta > 0$ , the inequality can also be written as:

$$\sin \theta + \sqrt{3} \cos \theta \leq 2.$$

Then using

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - t}{2}}, \text{ and}$$

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + t}{2}},$$

where we set  $t = \cos 2\theta \in [0, 1]$ , from

$$\sin \theta + \sqrt{3} \cos \theta \leq 2.$$

we get

$$\sqrt{\frac{1 - t}{2}} + \sqrt{3} \sqrt{\frac{1 + t}{2}} \leq 2.$$

After squaring and rearrangement, we get

$$t + \sqrt{3} \sqrt{1 - t^2} \leq 2 \quad \text{or} \quad \sqrt{3} \sqrt{1 - t^2} \leq 2 - t.$$

Squaring again we gain

$$1 - 4t + 4t^2 = (1 - 2t)^2 \geq 0.$$

This prove the inequality

$$\sin \theta + \sqrt{3} \cos \theta \leq 2.$$

Then the minimum of  $f(\theta)$  is obtained for:

$$t = \cos 2\theta = 1/2, \text{ that is, } 2\theta = 60^\circ.$$

Hence, the value of the angle  $\theta$  is  $30^\circ$ . By this The length of the minimal road network is:

$$S(30^\circ) = 2 + 2f(30^\circ) = 2 + 2\sqrt{3} \approx 5.4641.$$

**10** prove that sequences  $(a_n)$  and  $(b_n)$  satisfy the recurrence relations  
 $a_n = a_{n-1} + 2b_{n-1}, b_n = a_{n-1} + b_{n-1}, a_1 = b_1 = 1.$

**Proof**

$$\begin{aligned} a_n + b_n \sqrt{2} &= (1 + \sqrt{2})^n = (1 + \sqrt{2})(1 + \sqrt{2})^{n-1} \\ &= (1 + \sqrt{2})(a_{n-1} + b_{n-1} \sqrt{2}) \\ &= a_{n-1} + 2b_{n-1} + (a_{n-1} + b_{n-1})\sqrt{2}, \end{aligned}$$

**11** Determine which of the numbers  $\pi^e$  and  $e^\pi$  is the larger.

**Solution**

Consider the logarithms of the two numbers

$$e \ln \pi \text{ and } \pi \ln e = \pi.$$

Since  $\pi > e$  and the function

$f(x) = \frac{\ln x}{x}$  is decreasing on the interval  $[e, +\infty)$ , it follows that;

$$f(e) > f(\pi), \text{ that is, } \frac{1}{e} > \frac{\ln \pi}{\pi},$$

whence  $\pi > e \ln \pi$ . Thus  $e^\pi > \pi^e$ .

By these results, generally we have:

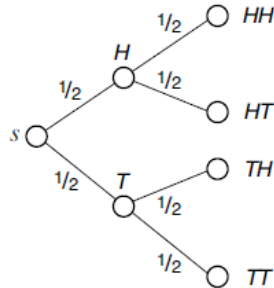
$$\frac{\ln a}{a} < \frac{\ln b}{b}, \text{ or } b \ln a < a \ln b, \text{ whence } a^b < b^a.$$

**12** Calculate the Probability of T and then H in Two Coin Tosses.

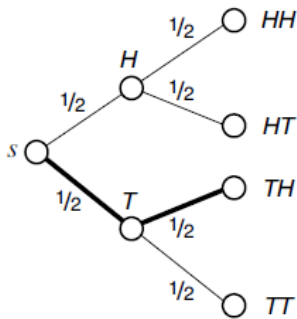
### **Solution**

In a decision tree, the probability of an outcome can be found by multiplying the edge probabilities along the path joining  $s$  and that outcome. Each outcome (which corresponds to a path from  $s$  to a particular endpoint) has a probability that can be calculated by multiplying the probabilities of the edges along the path. For

example, the path  $s \rightarrow H \rightarrow HT$  has probability  $1/2 \times 1/2 = 1/4$ ; in other words  $\Pr[HT] = 1/4$ . This is shown in the figure bellow:



This is also called weighted decision tree. Here we Use the decision tree in Figure above to determine the probability that **T** appears on the first toss and then **H** appears on the second toss of a coin. Therefore we have the associated path  $s \rightarrow T \rightarrow TH$ . This path has probability  $\Pr[TH] = \Pr[T \text{ and } H] = 1/2 \times 1/2 = 1/4$  as bellow:



**13** Prove that if  $n$  is a positive integer, then  $n^2 + 3n + 2$  is even.

**Proof**

If not, then  $n^2 + 3n + 2$  is odd for some  $n$ . Any odd number has the form  $2m + 1$  for some integer  $m$ . Hence

$$n^2 + 3n + 2 = 2m + 1.$$

But then:

$$n^2 + 3n - 2m = -1, \text{ or } n(n + 3) - 2m = -1.$$

Now, if  $n$  is even, then  $n+3$  is odd and if  $n$  is odd, then  $n+3$  is even. In either case,  $n(n + 3)$  will be the product of an even and an odd number and will thus be even. So  $n(n + 3) = 2k$  for some integer  $k$ . As a result we have:

$$2k - 2m = -1 \text{ or } 2(k - m) = -1.$$

But this shows that the number  $-1$  is even, and that is impossible.

We conclude that our initial hypothesis is false:  $n^2 + 3n + 2$  cannot be odd; it must be even.

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## **BOOK TWO**

## **PHYSICS**

"We are slowed down sound and light waves, a walking bundle of frequencies tuned into the cosmos. We are souls dressed up in sacred biochemical garments and our bodies are instruments through which our souls play their music".

**Albert Einstein**

**Physics thinking**

Thinking and reasoning to gain a general theorem is a process in mathematics while in physics we are seeking solutions to our problems with nature and natural phenomena. In fact we add something on the mathematical thinking and that is problem solving in theoretical and real nature. Hence thinking process in physics can be rigorous or not. In rigorous you start your problem solving process with the foundations of mathematics such as set theory and try to build a logical base for your way to solve your problem but another view is, mathematics is a tool to solve physical problems and for every problem we face, we need its appropriate mathematical tool to solve.

Ancient nations, had not a lot of contributions to the science however they really tried hard. Because they did not developed powerful mathematics we developed today. Therefore their understanding of universe did not travel so far.

The power of reasoning in physics gives us scientific procedures and approaches while the importance of understanding is very essential for finding your suitable mathematical tool to solve your physics problem.

Elementary characteristic of physical theories along with the laws that describe numerous natural phenomena are resulted from our intellectual conquest on different battle field with demons of nature. Nature is a field of relationships and causes which are different in image but similar in essence. Because of this, one can apply one tool for a lot of different problems. Since nature works on this chain of **relations**, one can build its way to solve a problem based on foundations of mathematics. Relations happens among sets, spaces or objects, interrelated with each other, so then sets, subsets, Cartesian Products and dimensions, therefore, functions and these functions changes and move since nature is in motion, then we need analysis and calculus, but if we live in space then we deal with geometry, Euclidean or Non-Euclidean, real space or complex space and their geometries. Consequently, having an elemental understanding of universe need a deep understanding of foundation of mathematics. May be this is the reason why universe speak with us by the language of mathematics.

It is fact that observation and experiment has a central role in modern science but remember that lots of people worked on the electromagnetism by observation and experiment then they made even precise interpretation but it was mathematician J.C Maxwell who formulate law of electromagnetism in simple mathematical formula. So with no exception physics is mathematics in practice in terms of our nature from atom to galaxy.

Instead of philosophizing to clarify our discussion let's do some concrete samples.

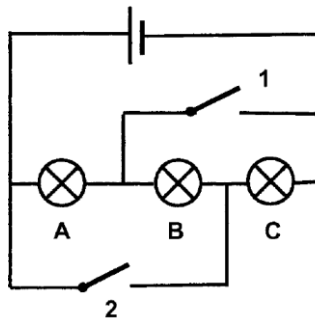
**1** Ice cubes float in a glass, filled to the brim with water. Will the water spill over when the ice melts?

The water does not spill over. According to Archimedes' principle a floating object displaces water that has a weight equal to that of the object. We can imagine that the floating ice cube has "dug a hole" at the surface of the water. The hole is precisely large enough to hold water with the same mass as the ice cube. Therefore the level of the water in the glass will not change when the ice melts.

**2** You ride an ordinary bicycle, in ordinary gear. Are there any points on the bicycle that don't move forward relative to the road?

Generally yes, the contact points between the tires and the road. Where the wheels touch the road, their speed relative to the road is zero since while their opposite parts move forward relative to the ground the tiers moves backward.

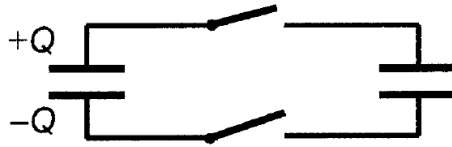
**3** A network has three equal lamps, connected as in figure.



When switch 1 is closed, lamp A lights up with normal brightness. Are there any lamps that light with normal brightness if switch 2 is also closed?

All three lamps have normal brightness when switches 1 and 2 are closed, because they are all coupled in parallel. Each lamp has one side directly connected with one of the battery poles, and the other side of the lamp is directly connected with the other battery pole. This solution assumes that the internal resistance of the battery is so small that the voltage does not depend on whether one or three lamps are connected.

4 You have two capacitors with equal capacitance  $C$ . One is given the charge  $Q$  and the other is uncharged. They are connected as in figure bellow.



When the two switches are closed, the charge  $Q$  will be equally divided between the capacitors. We know that the formula for the energy of a capacitor is:

$$E = \frac{Q^2}{2C}$$

But since this is also the total energy of the two capacitors before the connection is made, when the switches are closed, the charge  $Q$  becomes equally divided between the capacitors. Then the total energy is:

$$\frac{(Q/2)^2}{2C} + \frac{(Q/2)^2}{2C} = \frac{Q^2}{4C}$$

Half of the energy that was originally stored in the charged capacitor seems to have disappeared. Where has it gone?

The missing energy converted heat in the connecting wires, because they cannot have a resistance that is precisely zero, or it can be converted to joule heat, radiated out or other ways of dissipating energy in a system like this. The relative importance of various loss mechanisms depends on details in the circuit and on how it is closed.

**5** Even in the absence of centrifugal effects from the Earth's rotation, the surface of the oceans would not have perfect spherical shape, because the mass distribution in the Earth does not have perfect spherical symmetry. Consider an underwater mountain, protruding from an otherwise almost flat seabed. Is the ocean surface above that mountain slightly depressed or does it form a small hump?

The gravitational force between two bodies only depends on their masses and not on what they are made of or whether they are solid or liquid. If the underwater mountain is replaced by water, the ocean surface would be flat (neglecting the curvature of the Earth). Because minerals have a higher density than water, the mountain pulls with an increased gravitational force on the ocean

water. That force must be perpendicular to the ocean surface, since any force component parallel to the surface would set the water in motion. Therefore the extra gravitational force from an underwater mountain gives rise to a small hump on the ocean surface.

**6** A mad driver is moving at the rate of 35 m/s on a residential street! A cat suddenly jumps in front of the car 105 m away. The driver slams on the brakes “immediately” with a reaction time delay of 0.2 second. The brakes cause a deceleration of 6 m/s<sup>2</sup>. Is the cat dead or alive?

The reaction time is the time that it takes for the act of seeing the cat in the middle of the street to translate into applying brakes, which in this case is 0.2 second. During this time the car travels uniformly with its initial speed. Thus, the distance in this case is:

$$x = vt = 35 \times 0.2 = 7 \text{ m}$$

The equation of velocity as a function of time is:

$$v(t) = v_0 + at$$

And the distance turns out to be:

$$x(t) = v_0 t + \frac{1}{2}at^2,$$

Here

With  $v_0 = 35$  and  $a = -6$ ,

We have

$$v(t) = 35 - 6t$$

$$x(t) = 35t - 3t^2.$$

By these two equations we have the speed and distance traveled for any given time  $t$  in this particular problem.

For the car to come to a complete stop,  $v$  has to be zero. Therefore;

$$0 = 35 - 6t \Rightarrow 6t = 35 \Rightarrow t = \frac{35}{6} = 5.833 \text{ s}$$

Substitute this  $t$  in the equation for  $x(t)$  to find the distance:

$$x = 35 \times 5.833 - 3(5.833)^2 = 102 \text{ m.}$$

The total distance is the sum of the two distances found as:

$$x_{\text{tot}} = 7 + 102 = 109 \text{ m.}$$

Therefore cat is dead! Since the driver is designated as “mad” can be appreciated by converting the speed to mph:

$$35 \text{ m/s} = \frac{35}{0.4472} = 78.26 \text{ mph!}$$

With these samples we see that, there is a need to know the evolution of ideas in physics and their rules to solve more complicated problems we face.

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## Physics Story

The story of physics is tightly fastened to mathematics, and geometry in particular. Therefore it started from ancient Mesopotamia and Egypt then comes to Greece then to Muslims then early European and their scientific revolutions. This comes to Newton, Hamilton, Boltzmann...to Einstein and Hawking then goes on now and forever.

While the Mesopotamians used arithmetic and a kind of algebra to investigate nature, the Greeks like the Egyptians used geometry. The Mesopotamian focused on prediction while the Greek was often more concerned with explanation than prediction. Exploring Mesopotamian science is on the way and people around the world know a little about it. But the Greek science is known well consequently we start our story with them.

They tried to explain natural phenomena through a system of rational reasoning opposed to Mesopotamian and Egyptian astrological causation and reasoning. This system was called *natural philosophy*. They even proposed the revolution of some planets and earth around the Sun which led to the old heliocentric model just by conjectures.

Greek mathematics and philosophy, began with Thales of Miletus (640–546 B.C). He was, the first of a long line of philosopher-mathematicians. By the clever use of ratios, many generations of Greek mathematicians, were able to make extraordinary discoveries about the universe. In Egypt once Thales wished to measure the height of Giza pyramid he found that the length of the shadow of the pyramid equaled the height of the pyramid in this way.

Aristarchus (310–230 B.C) used ratios and angles to estimate the relative distances of the Earth to the Moon and the Earth to the Sun. His reasoning was on the *quantitative observation* of the heavenly bodies. From the shadow of the Earth on the Moon in a lunar eclipse, he could estimate the diameter of the Moon to be  $\frac{1}{3}$  of the Earth's diameter. Having found the size of the Moon, he found the Earth-Moon distance to be 25 Earth diameters by measuring the angular size of the Moon. Hipparchus improved his estimate to 30 Earth diameters, which is very close to today's value.

Mesopotamians and Greeks mathematical science development had no job with

fundamental concepts of *mass*, *force*, and *energy*. However Archimedes is an exception. Therefore we got him and then to Middle Ages. Because Without mathematics it is hard to separate useful ideas from useless ones.

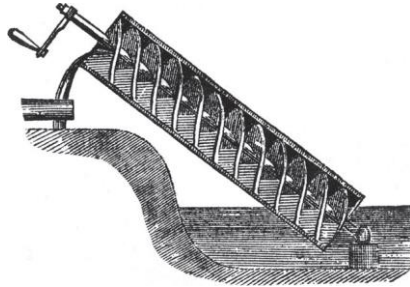
Among these great men of elementary science was the first and one of the greatest physicists and mathematician of all time Archimedes of Syracuse (287–212 BC). He spent most of his life at Syracuse working on levers and other aspects of mechanics. In hydrostatics he devised a pump called *Archimedian screw* and formulated *Archimedes' principle*. His method of successive approximations called the method of exhaustion similar to integration we use today allowed him to determine the value of  $\pi$  to a good approximation. He was killed by a soldier in the Roman invasion to Syracuse.

Archimedes' principle says "The weight of the liquid displaced by a floating body is equal to the weight of the body. The principle was not in fact stated by Archimedes, though it has some connection with his discoveries. The principle is often stated in the form: when a body is (partially or totally) immersed in a fluid, the upward thrust (**buoyancy**) on the body is equal to the weight of fluid displaced

The thrust on a body immersed in a fluid. This can be written in a compact expression as:

$$B = w_f,$$

Where  $B$  is the buoyant force and  $w_f$  is the weight of the fluid that the object displaces.



Above is a sample of Archimedean screw. The intertwined helical blades with the bottom end of the screw is immersed in a pond along with the rotation of the screw raises water from the pond to a higher elevation.

After ending the flourishing Greek movement in science around the 1<sup>st</sup> century AD, the dark ages started in western culture. However it was flourishing in the Islamic world. "Most of the treatises written by Indian, Assyrian, Persians and, above all, ancient Greek natural philosophers were translated into Arabic". This enabled

Islamic scholars, to have an Islamic Golden Age during the 750-1250 AD.

From the point view of physics, two of these numerous important Islamic scholars of this period are prominent.

Avicenna (Abu Ali ibn Sina) 980- 1037 AD was a Persian polymath. "He wrote more than 400 treatises on a wide range of subjects, including philosophy, physics, astronomy, alchemy, geology, and mathematics, from which 240 have survived". His most important works are *The Book of Healing* and *The Canon of Medicine*. The first is a broad philosophical and scientific encyclopedia, thus a healing for mind and soul. The second, was a standard medical textbook in many medieval universities. It was *Canon (rule)*, since it was based on anatomy and structure of human body.

Alhazen (Abu Ali ibn Haytham) 965-1040 AD also was a polymath of Persian origin. From his more than 200 treatises only 55 have survived. "He made significant contributions in the fields of mathematics, optics and astronomy". He was known as "Ptolemy the Second".

After dark ages and Muslims most philosophical interpretation of universe it was in the 14<sup>th</sup> century which the road of ideas about mathematics and the physical sciences began to turn to light with persons like French mathematician Nicholas Oresme (1325–1382). His most famous contributions are graphical analysis of motion under constant acceleration or deceleration and innovation treatment of infinite series and harmonic series in particular.

IN this transition time we also have Polish astronomer, Nicolaus Copernicus (1473–1543). When most of the European scholars believed that Earth is situated at the center of the universe from Ptolemy, Copernicus believed in heliocentric model of early ancient Greek. His writings had revolutionary effects on the science and philosophy. His major work, *De Revolutionibus Orbium Coelestium* (On the revolutions of the celestial spheres), formed the basis of what is now known as the Copernican revolution. He did not spend much his time systematically observing the heavens himself and worked more mathematically. Because of the lack of accurate observation his ideas, like Ptolemy's,

are still geometric and not physical so then his revolutionary ideas did not convince many people when the book was first published.

German astronomer, mathematician, and physicist Johannes Kepler (1571–1630) began his studies with Copernicus's model with data collected by the Danish astronomer Tycho Brahe (1546–1601). Brahe and his staff made an extraordinary number of measurements of the positions of all known planets plus hundreds of stars.

Using these measurements Kepler discovered three laws that described the elliptical motion of the planets in space instead of old prevailing perfect circular orbits. Approximately fifty years later, Newton showed that Kepler's laws are a direct consequence of motion under the influence of gravity.

Kepler's three laws of planetary motion are:

**First Law:** Each planet moves around the Sun in an elliptical orbit with the Sun located at one of its foci. Mathematically as:

$$R_{\text{aphelion}} = a(1 + e), R_{\text{perihelion}} = a(1 - e)$$

Where  $R$  is distance. Aphelion is the farthest distance and perihelion is the closest distance of the planet with the sun.  $a$  is the length of the ellipse's *semi major axis*, and  $e$  is the *eccentricity* of the ellipse.

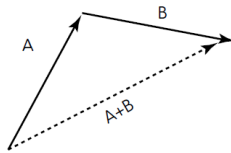
**Second Law:** The line joining the Sun to the planet sweeps out equal areas of the ellipse in equal times. Hence, the planet moves faster when it is closer to Sun.

**Third Law:** The Square of the period  $T$  of revolution of each planet is proportional to the cube of its semi-major axis  $a$ . In symbols this is written as  $T^2 = ka^3$ , where  $k$  is the constant of proportionality.

Almost all of the scientists and mathematicians until now were primarily interested in developing geometric descriptions of nature. However, Kepler showed the importance of observation, they were, Flemish Simon Stevin (1548–1620) and the Italian Galileo Galilei (1564–1642) who changed the way of physics and natural sciences fundamentally by introducing experiments. They founded a careful combination of experimental science and rigorous mathematical modeling

which characterizes many aspects sciences. Therefore all scientific ideas were open to scrutiny and *reproducible* experimental testing by anyone with sufficient knowledge, equipment, and technique to devise and perform the necessary experiments.

Stevin's Idea of decimalization of the number system paved the way for real number system. He also invented an arithmetic law of forces to compute the effect of two forces acting at a point.



He is best remembered for these discoveries and also in *statics* that branch of physics that deals with the forces, in bodies at rest. Similar to importance of point, line, and plane in Euclidean geometry, forces are fundamental to Stevin's concept of physics.

As a student at the University of Pisa in 1583, Galileo Galilei discovered the constancy of the period of a pendulum while watching the oscillations of a lamp at the cathedral of Pisa. To

support himself, he gave private lessons in mathematics. Following Archimedes, similar to Stevin he introduced experimental procedures in the study of motion. He is the first observed the night sky through a telescope. Rigorous mathematics and careful designed experiments were characteristics of Galileo's works.

His two major works, are "Dialogue Concerning the Two Chief World Systems (Ptolemaic and Copernican) and Dialogue Concerning Two New Sciences. They are called dialogue since they are written in the form of conversations among characters.

Complete treatment of the problem of motion, and development of the essential framework to examine motion are among his contributions. However there were many problems remained unsolved since he did not know sufficient math that did not yet exist. Thus there was a need to a revolution in mathematics since progress in science sometimes depends on progress in mathematics.

During the Renaissance Kepler discovered an elliptical Planetary orbits and Galileo discovered

that in the absence of air resistance, a projectile follows a parabolic path as some brand new usage for curves than Greeks and their Islamic successors. Although many new curves were discovered during this time as well. French philosopher, mathematician, and scientist René Descartes (1596–1650) and the French lawyer and mathematician Pierre de Fermat (1601–1665) had great contributions to the foundations of new revolutionary approach in mathematics. They connected algebraic equations with exactly two unknowns to numerous geometric curves. The responsible was a new type of geometry called *Coordinate geometry*. With this type of geometry there were infinitely many different types of curves to study. Their study also needed new types of mathematical tools such as calculus and linear algebra.

Isaac Newton (1642-1727) had a very rough childhood but He was a quiet boy with a very active imagination. His father died before his birth; his mother remarried and left him with his grandmother on a farm. His school reports described him as "idle" and "inattentive." His mother, returned with reasonable wealth and

property. She thought that her eldest son was the right person to manage her affairs and her estate. Newton's management was disappointing thus Newton's uncle persuaded his mother that Isaac should finish his schooling and go to the university. He entered Trinity College, Cambridge.

At first he interested in chemistry which remain with him until the end of his life. He found a copy of Euclid's *Elements* and read it. He also studied the works of Viète, Kepler, Galileo, Fermat, Huygens and others. In the summer of 1665, he had to leave Cambridge due to the widespread plague. He spend 18 months in his family farm at Woolsthorpe. During this time he established calculus, discovered the laws of motion, and found a mathematical definition and formulation of gravity.

Newton's discovery of *fundamental theorem calculus* is so important since allowed mathematicians to have a reverse back process and find the unknown function when derivative of a function is known. This is the birth of new branch of mathematics called *differential equations*. The laws of nature are generally

expressed in terms of differential equations. Because it has inhomogenous motions thus has first order to higher order derivatives such as velocity, acceleration and jerk originated from a function of series of functions.

At the same time German mathematician and philosopher Gottfried Wilhelm Leibniz (1646–1716) was working on the subject. Unlike Newton he was the first to publish his ideas in 1684. He was a true genius and a child prodigy such that he received his doctorate degree when he was 20 years old. He devised a mechanical calculator, helped to the development of logic and algebra, and discovered the base 2 number system, which is very central in computers. The symbols we use today in calculus were introduced by him.

Newton's most contributions were in the mathematical study of motion. In "*Philosophiae Naturalis Principia Mathematica*", he used an axiomatic approach alike Euclid elements to expose his ideas. He stated his three laws of motion as:

1. Anybody continues in its state of rest, or in uniform (constant) motion in a straight line, unless it is compelled (forced) to change that state by forces impressed upon it.
2. The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.
3. To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always directed to contrary parts.

The first law states that in the absence of forces a body can be at rest or it moves at a constant velocity along a straight line. Newton's second law describe, the ways that forces affect motions, using vector property of a force, first described by Stevin. The third law states the idea of conservation of momentum provided that, forces occur in pairs, with opposite direction but equal in magnitude. Consequently "when we add the two forces together they cancel each other, or add up to 0".

French mathematician Pierre-Simon Laplace, (1749–1827) is best known for his work on a model of planetary motion based on Newton's gravitational theory, to compute the orbital perturbations of planets due to gravitational interaction with each other. He described his ideas in a five-volume book *Mécanique celeste*. He's also remembered for fundamental contributions to the theory of probability and deterministic view of closed dynamical systems.

Until this time Galileo, Newton, Laplace, Verrier and others had developed our understanding of nature. The discovery of Neptune by Le Verrier happened with the help of new mathematical tools such as Laplace planetary model. After discovery of Uranus astronomer found that another force affecting Uranus's motions. It was additional force from the gravitational attraction of Neptune. Therefore Le Verrier had discovered the cause if perturbations in Uranus orbital motions i.e. the planet Neptune, by principle of the *conservation of momentum*, measurements of Uranus's motion, and a great deal of mathematics.

Thus until now we said that:

Galileo "founded the branch of mechanics called *kinematics* and confirmed the *heliocentric* theory". He also described the transformation rules for the position and velocity of a body in a reference system, when it is changing its frame of reference as:

$$x' = x - v_0 t \qquad v' = v - v_0.$$

Using this method he examined the motion of a particle under constant velocity and acceleration. Due to the lack of infinitesimal methods (calculus) in mathematics he just solved kinematical problems using basic analytical geometry.

Then we came to Leibniz symbolizing of calculus and Newton's theory of mechanics which was based on three axioms. The first two laws can be combined in the vector equation of motion of a point mass in a famous form as:

$$\mathbf{F} = m \cdot \mathbf{a}$$

Where  $m$  is the *inertial mass* of the particle and constant of proportionality as well,  $\mathbf{a}$  is the acceleration and  $\mathbf{F}$  is the total force acting on the particle. Or more precisely in differential form as:

$$d\mathbf{p}/dt = d(m \cdot \mathbf{v})/dt = m(d\mathbf{v}/dt) = m \cdot \mathbf{a}$$

The middle equality is showing that this relation is true when the mass of the body remains constant. It also showing that force is equal to infinitesimal changes of momentum with respect to time.

Then Newton calculated differentials (infinitesimal changes) of various forms of force. This then called differential equations of motion. He proved Kepler's elliptical planetary motion and based on these he stated universal gravitational law as:

"Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses, and inversely proportional to the square of the distance between their centers of mass".

$$F = \frac{Gm_1m_2}{r^2} \quad \text{and} \quad F = \frac{GMm}{r^2},$$

Where the second equation is usually used when a celestial body of mass  $M$  influences the motion of an ordinary object of mass  $m$ .

But what is  $G$ ?

It is a constant of proportionality called the *universal gravitational constant*. "Its value is dependent on the decision to arrange for the *gravitational mass* and the *inertial mass* of a particle to have the same value. The dimensions of  $G$  are  $L^3 M^{-1} T^{-2}$ , and its value is  $6.672 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$ ." British physicist Henry Cavendish measured it in 1798. To measure  $G$ , one had to measure the two masses  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , the force  $\mathbf{F}$ , and the distance  $\mathbf{r}$  between them based on the fact that, "the acceleration due to gravity is independent of the mass of the object being accelerated".

We also noted that, they were, Le Verrier and Laplace how extended Newton works to a more general point view. Now we continue to our story in *optics* with stick in mind that;

Physical theory is just a mathematical model and that it is meaningless to ask whether it corresponds to reality.  
*Hawking*

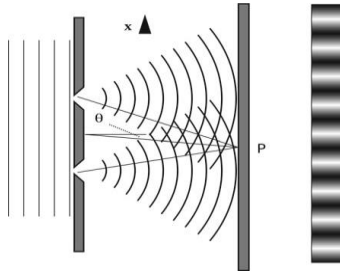
According to Oxford dictionary of physics, Optics is:

"The study of light and the phenomena associated with its generation, transmission, and detection. In a broader sense, optics includes all the phenomena associated with

infrared and ultraviolet radiation. *Geometrical optics* assumes that light travels in straight lines and is concerned with the laws controlling the reflection and refraction of rays of light. *Physical optics* deals with phenomena that depend on the wave nature of light, e.g. diffraction, interference, and polarization".

Although Greeks and Islamic scholars had done lots of geometrical interpretations in optics, they were Newton and Christian Huygens (1629–1695) who made great fundamental contributions to the natural behavior of light. Newton published his results in *Optick* in 1704. He believed that light is "a beam of tiny particles streaming through space". However Huygens, believed that light has a wave nature. In fact, even Newton experiments showed the wave behavior of light. Although we mustn't forget Willebrord Snell from Netherlands. He was contemporary with Tycho Brahe and Johannes Kepler. In 1601 he stated that; "the angle of *refraction* of light that travels between two media depends on the refractive indices of the media". This is called *Snell's Law*. Refraction occurs when a ray or bundle of light passes from one medium to another if they have different refractive indices i.e. different amount of changing in velocity of

light when it passes through them. It was British physicist Thomas Young (1773–1829) finally demonstrated the wave nature of light, with his famous double-slit experiment as bellow:



The interference of waves accrues since we have two cuts that light propagate through them. If we have one slit then this will not happen. Observing this phenomenon Young assumed that light consists of *transverse waves* which *oscillate* in directions perpendicular to each other with no destructive interference so no zero amplitude. This assumption was correct but its mathematical foundations and also for wave optics were established by French physicist Augustin Jean Fresnel (1788–1827). He formulated the mathematical wave theory of light to its final form, selecting the problem of "diffraction of light by a circular opaque disk". His results,

proved that light is a wave phenomenon. Although in the next coming years, it was demonstrated that light has a double behavior of wave and particle.

Using Newton physics in different media, in the late 18<sup>th</sup> and early 19<sup>th</sup> century, it was understood that *mechanical waves* appear in many various situations. It is found that, except light, all other waves such as, sound, wind and water etc. follow the rules of mechanical waves in an elastic medium. Thus, there was a need for universal wave equation which expressed as bellow:

$$\nabla^2 \xi = \frac{1}{V^2} \frac{\partial^2 \xi}{\partial t^2},$$

Where  $\nabla^2$  is the Laplacian operator that represents how  $\xi$  changes in space,  $\xi$  represents the displacement of the medium from its equilibrium position,  $\frac{\partial}{\partial t}$  how  $\xi$  changes with time and  $V$  is the speed of wave which for light is  $C$ . Therefore  $\xi$  is a function of space and time. Keep in mind that mechanical waves produced in elastic media by *mechanical oscillations* and the universe is recognizable for us due to existence of

*waves and vibrations* in different phenomena and matters.

Newton showed the conservation of momentum. The conservation in matter was demonstrated by French chemist Antoine-Laurent Lavoisier (1743–94) which then is called *conservation of mass*. Instead of qualitative approach in chemistry he developed a new quantitative method by precise measurements. It was him who investigated the role of air in combustion and distinguished between chemical elements and compounds in contrast with Greek's four elements: earth, air, fire, and water. His ideas was; common measure of matter is *weight* against Italian polymath Leonardo da Vinci (1452–1519) of *volume*. Due to different *compressibility* in different matters such as water and gas he assumed that volume cannot be a common measure but at surface of planet Earth which mass and weight are proportional to each other weight is a more fundamental measure than volume. We use *mass* as "a measure of the quantity of matter in a body". Due to these facts, we can compute the mass since we can measure the weight.

Using a chemical model of reactions among different matters before and after reaction, he concluded that, in an isolated system the mass of the system is constant under all transformations thus matter can neither be created nor destroyed. This is the law of *conservation of mass* (matter). Mathematically:

$$\frac{dm}{dt} = \dot{m} = 0$$

For a non-isolated system, we must compute the difference between the initial amounts of mass in a phase of a system with the terminal amount to have the amount of mass changes in a system. With generalization of this kind we have *deterministic differential equations* to approximate every quantity in a system with respect to time or another independent variable. Consequently, we can predict the future of system, having its initial data, function, derivatives (rate of changes with respect to time) of the function which satisfy related differential equation. But how about a set of differential equations?

Swiss mathematician Leonhard Euler (1707–1783) studied above discussion for both conservations of momentum and mass in a phenomenon. He produced "a set of differential equations" to describe the motion in a system especially fluids for the first time. He did this with the set of axioms in the same way as Euclid for geometry and Newton for rigid bodies. Therefore he showed that one can consider fluids as a "deductive discipline" so then mathematically.

However it was Christian Huygens who proposed the concept of *moment of inertia*, which plays a role similar to that of mass in the equation of the *rotational motion* of a rigid body. He also formulated the laws that describe the collision of two bodies.

In these period we also have a family of brilliant Swiss mathematicians from Basle. The most famous of them are Jacques, Jean and Daniel Bernoulli. Jacques Bernoulli (1654–1705) developed calculus, and probability theory. Jean Bernoulli (1667–1748) proceeded calculus of variations and discovered *l'Hôpital's rule* as well. He also proposed the *brachistochrone* (the shortest curved path from point A vertical to B

horizontal) problem. Daniel Bernoulli (1700–1782) remembered for his law of fluid dynamics or hydrodynamics. It says that:

"The total energy of fluid pressure, gravitational potential energy, and kinetic energy of a moving fluid remains constant. For liquid flowing in a pipe, an increase in velocity occurs simultaneously with decrease in pressure". Mathematically say;

$$\frac{v^2}{2} + gz + \frac{p}{\rho} = C,$$

V: the fluid velocity,

G: the acceleration due to gravity,

Z: the elevation (height) of a point in the fluid,

P: the pressure,

$\rho$ : the fluid density,

C: is a constant.

Thus, in a moving body, not only kinetic energy changes due to potential energy as well as the body elevate but also due to pressure. This formula will respond in a steady, incompressible, non-viscous (without internal friction) fluid flow in a closed pipe. Lots of restriction! Why? Since in real situation, even for an apparently simple system, the states of that system appeared to be

very complex, most of laws in physics designed for idealized situations.

Johann Heinrich Lambert, (1728–1777) was a Swiss–German mathematician, scientist and philosopher. He showed that  $\pi$  is irrational, using continued fractions, presented a standard notation for hyperbolic functions. In physics we have *Lambert's Law of Emission* which says:

"The intensity (flux per unit solid angle) emitted in any direction from a region of a diffuse (slightly rough) surface is proportional to the cosine of the angle between the direction of radiation and the normal to the surface". Mathematically;

$$I_e \propto \cos \theta,$$

Lambert and Immanuel Kant (1724–1804) considered universe as a collection of *fractal*, or *hierarchical* structures, which we call *galaxies* today.

Although until the end of 17<sup>th</sup> century Newton's mechanics had got a maturity in its foundations, there existed some restrictions yet. Newtonian mechanics largely depended on concept of "force"(internal and external) and it was uncooperative in changing degrees of freedom

and dimensions. We need some other to remove these restrictions as follows:

A French-Italian mathematician Joseph Lagrange (1736–1813) presented two new dynamical theories. A French engineer and mathematician, Joseph Fourier (1768–1830), had fundamental contributions to the theory of *heat conduction*. He also studied trigonometric series which brought to us the so-called *Fourier series* which has huge importance in physics, engineering and also mathematics itself. Totally we have four French mathematicians; Pierre-Simon Laplace (1749–1827), Adrien-Marie Legendre (1752–1833), Joseph Fourier (1768–1830) and Joseph-Louis Lagrange, (1736–1813) in the period of the French Revolution who had great contributions to calculus and mechanics. We remember, Legendre polynomials and his solutions of a certain differential equation, and also Lagrange's *Mécanique analytique*, published in 1788. Lagrange is also known for his results in number theory, algebra, probability and theoretical mechanics. So may be the nearest figure to Euler. Generally they developed classical mechanics in higher level to overcome

above mentioned restrictions by introducing some generalizations in coordinates, state and energy of a system. Thus they developed an elegant way in which the Newton laws of mechanics could be applied to complex situations.

In this time other fields of physics were not developed as far as mechanics had done. It was Rumford (1753-1814) who proposed the connection between heat and motion in 1798. Although the quantitative study of heat and its synthesis with the other branches of physics took place in 19<sup>th</sup> century. Until this time Geometrical interpretation of optics had progressed, high quality lenses and mirrors manufactured, refracting telescopes developed but the study of *diffraction* and *polarization* were in their basic steps.

Electricity and magnetism and their relationships were largely unknown and qualitative. In gravitational force we had an inverse square law from very basics. This is also true for *static electric charges*. It states that: the force between static electric charges varies as the inverse square of the distance between them, just as does the between two masses. Coulomb (1736-1806) is remembered for this experimentally law. Galvani

(1737-1798) and Volta (1745-1827), with their battery (electric pile) had opened the way for the examination of electricity in motion or electrodynamics. *Law of the conservation of energy*, the *wave theory of light*, and the *atomic theory of matter* were true child of physics in 19<sup>th</sup> century. Based on Rumford and of Davy (1778-1829) discoveries, Joule (1818-1889) measured the *mechanical equivalent of heat*. This paved the way for the *law of the conservation of energy* which should be added to the previous conservation laws, of *mass*, *momentum*, and *angular momentum*. The law on conservation of energy, following generalization process in formulation, sum up two separate conservation laws of mass and momentum in one. It states that:

$$E_k + E_p = \text{constant},$$

$E_k$ : Kinetic energy and  $E_p$ : potential energy. The energy equation follows by integration with respect to time ( $t$ ).

Based on these laws and works of Carnot (1796-1832), *thermodynamics* was born. This is done by Kelvin (1824-1907), Helmholtz (1821-1894), Clausius (1822-1888), Gibbs (1790-1861), among others. They showed that thermodynamics, is powerful tool applicable to all phenomena. They proposed *first and second, third and zero laws of thermodynamics*.

Quantities in thermodynamics are macroscopic. They determine the internal state of a system, and thus its internal energy ( $E_I$ ).

Greeks and Islamic scientist were aware of atomic basic of matter. Although it was Dalton (1766-1844) realized that:

"Each chemical element or compound came in the form of large numbers of individual particles, as smallest bit of matter which had the chemical characteristics of the material".

This lead to great simplification in chemistry and laws of gas behaviors discovered by Boyle (1621-1679). Avogadro (1776-1856), proposed that:

"The numbers of molecules in equal volumes of different gases (under the same conditions) are equal"

Finally it was Cannizzaro (1826-1910), who combined these ideas and develop the modern concepts of *atomic and molecular masses*.

In 1817 Oersted (1777-1851) discovered **magnetic field** of an electric current and Ampere (1775-1836) developed its mathematical relationship. This is known as *Ampere's law*. This law "connects the magnitude and direction of the magnetic field with the magnitude and

direction of the electric current and with the geometry of the system".

With these discoveries a question raised as:

*Could magnets produce electric currents?*

Henry (1797-1878) and by Faraday (1791-1876) answered yes. They showed that "*changing magnetic fields could produce an electric field, and so could induce currents in a conducting material*". Thus, Electricity and magnetism were connected.

From the heritages established by Euler, Lagrange, Laplace, Hamilton Fourier and Gauss (1777-1855) computation of gravitational and electric fields which surround masses or electric charges became very simplified. These became true by the development in methods of differential equations and vector analysis.

With these methods British mathematical physicist James Clerk Maxwell (1831-1879) gave a remarkable short mathematical description of:

- 1) "The inverse square law of force between electric charges at rest".
- 2) "A similar law for the force between *magnetic poles*".
- 3) "Ampere's law connecting the *magnetic field* in the neighborhood of a current with the current producing".
- 4) "*Faraday's law of induction*, which gave the electromotive force (voltage) induced in any circuit in

terms of the time rate of change of the *magnetic flux* through the circuit".

These are very fundamental to the field theory of electromagnetism which mathematically expressed as.

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Where  $\mathbf{B}$  is the magnetic flux density,  $\mathbf{D}$  is the electric displacement,  $\mathbf{E}$  is the electric field,  $\mathbf{H}$  is the magnetic field strength,  $\mathbf{J}$  is the current density,  $\rho$  is the free electric charge density, and  $t$  is time.

A symbol of hard working, Michael Faraday (1791-1867) proposed "lines of force" for electromagnetic field as an equivalent, to the concept of action at a distance for gravitational field. However he did not know mathematics he needed, thus it was Maxwell who proposed a "mathematical form in which two vector quantities, the electric field strength and the magnetic field strength, of the space surrounding the charges". He emphasized on the **field**, or "the

changes in the nature of space, produced by charges".

Towards the end of the nineteenth century the everyday application of these efforts were becoming more evident. For instance, there were two approach to produce and deliver electric power to consumers. Direct current approach of American inventor Thomas Edison (1847-1931), alternating current approach of Nikola Tesla (1856-1943), a Croatian-American engineer. The second was more efficient than the first since it reduced the dissipation of electricity to heat and then loss of current in final consumer, using a step-down transformer.

Thus there seemed to be a reciprocal relationship between electricity and magnetism, since induction needs a *changing* magnetic field whereas magnetic fields are produced by *steady* electric currents. Maxwell proposed that; "*a changing electric field produces the same magnetic effect as a current flowing in a conductor, that it constitutes a displacement current*".

Maxwell died in the age of 48. He did not see confirmation of his theory by Heinrich Rudolf Hertz (1857–1894) in his experimental discovery of electromagnetic waves. He know that

according to Maxwell theory the power of electromagnetic waves is a function of its frequency. Consequently, with some set of simple experiments he showed that, electromagnetic waves are reflected, refracted and polarized, just like light. Thus, Hertz's experiments confirmed Maxwell's theory such that light actually is a form of electromagnetic waves.

Adding the *displacement current* (time derivative of electric field) to the previous concepts, Maxwell predicts electromagnetic waves and proposed that:

"The electromagnetic (EM) waves travel at the speed of light. Furthermore, this speed does not depend on how the waves are produced".

Studying *polarization phenomena* shown that light was a *transverse wave* and measurements of its speed, exactly approved with the Maxwell prediction i.e.  $3 \times 10^{10} \text{ cm/s}$ .

From previous discussions we saw that there was an effort to interpret all branches of physics by laws of mechanics. Electromagnetism was not an exception. If we suppose that similar to mechanics it is "variation of force" that causes current and particle propagation with a finite speed then momentum and energy of the system are not be conserved since during its propagation they would lost. But we know that EM waves

propagate in a continuous medium called "field" thus the momentum and the energy from the one particle or current would be transferred to the other particle or current and this continues. So this inconsistency was solved by Maxwell's theory itself.

Another inconsistency was that the speed of light is the same in all frames of reference thus the *Galilean transformations* of space and time were not applicable for EM wave propagation. For this the works of Hertz, Hendrik Lorentz (1853–1928) and Henri Poincaré (1854–1912) introduced a radical idea to merge mechanics with electromagnetism. The idea says, since the velocity of EMW is constant and independent of the frame of reference, then "in a moving inertial reference frame the length of each material body contracts in the direction of motion, while the time intervals expand". This contraction

$(\sqrt{1 - \frac{v^2}{c^2}} : 1)$  is important at or near the speed of light. *Lorentz transformations* instead of *Galilean transformations* proposed a new rule of spacetime since they are "the equations for transforming the behavior of bodies in one frame of reference to another."

If  $x$  is taken in the direction of the relative velocity of the two frames, and  $v$  is the magnitude of the relative velocity then:

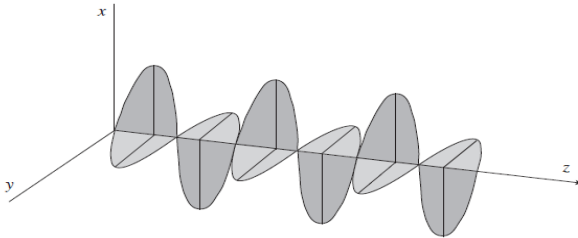
$$x' = \beta (x - vt), \quad y' = y, \quad z' = z$$

$$t' = \beta \left( t - \frac{vx}{c^2} \right)$$

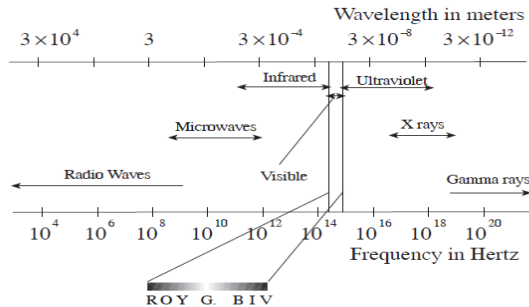
Where  $\beta = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$

Thus, time is not a universal constant and that space-time needs to be treated as a 4-D construct when high speeds are involved. Einstein's special theory of relativity and his theory of unification of space and time explained why these happens in universe.

Modern atomic theory of matter originated from electrodynamics. In fact this is a point of two ways in physics rooted in discussion between Newton and Huygens of particle or wave nature of light. This came to the hand of Maxwell and Planck to change our understanding of world. Thus let's go on with two figures.



This is a plane electromagnetic wave that has an electric field and a magnetic field. They are perpendicular to one another. The electric field oscillates along the x-axis. The magnetic field oscillates along the y-axis. The total EM wave is propagating in the positive z-direction.



This is EM wave spectrum from long and weak radio waves to short and strong Gamma rays. The intensity of every part depends on its wavelength. The shortest, the strongest, the longest the weakest with no limit in both sides. Our eyes can only detect after infrared and before ultraviolet.

Above figures importance is that they are building blocks of connection between thermodynamics, energy conservation, radioactivity of matter and electrodynamics to gift us quantum mechanics and General relativity, i.e. Modern physics against classical physics.

It was Hermann Helmholtz (1821-1894) who proposed the law of energy conservation as:

"There is a *finite constant* quantity, called energy, which can appear in various forms such as heat or mechanical work".

German mathematician and theoretical physicist Max Planck (1858-1947) was working on this subject. In his thesis he considered thermodynamic processes about "the relationship between energy and wavelength". His works provided elementary foundations of quantum mechanics. Until 1890, radioactive effects of some elements was discovered. It was supposed that these elements had a "supply of spontaneously generated energy". This bring the principle of the conservation of energy under question though. A German mathematician Emmy Noether (1882-1935), put forward a theorem of connection between time and energy conservation. Using new developed tools of abstract algebra such as commutative groups, Noether Theorem showed that "if time intervals are independent of their starting points then energy must be conserved". Thus energy, similar to mass and momentum (linear or angular) is conserved.

To build our story of starting quantum mechanics we need another guy. Austrian mathematician and physicist and philosopher Ludwig Boltzmann

(1844-1906), is the founder of statistical mechanics. He worked on the kinetic theory of gases.

Boltzmann had many scientific opponents partly because he introduced probability theory into physics. He had a soft-hearted personality, and a true igneous "far ahead of his time". Our today understanding of thermodynamics is hugely obliged to this great man of science. He discovered that:

The probability for a system to have a specific value  $E_s$  for energy is, proportional to Boltzmann factor as;

$$e^{-E_s/k_B T}$$

This factor is at the center of statistical mechanics and thermodynamics. Thus for system of interacting  $n$  particles we have energy  $E_n$ . In result,

$$P(E_n) \propto e^{E_n/K_B T}$$

If a particle assumed a system.

German physicist Gustav R. Kirchhoff (1824-1887) introduced the notion of *black body* for radiation due to heat. A complete black body can absorb all coming radiation. He formulated this as:

$$\frac{e}{a} = \Phi(\lambda, T)$$

Where e: emission, a: absorption,  $\lambda$ :wavelength and T: absolute temperature.

$\Phi(\lambda, T)$  is *spectral energy flux* since multiplying by the difference between two neighboring wavelengths ( $\Delta\lambda$ ), yield energy flux(energy amount crossing  $1\text{m}^2/\text{s}$ ) in that  $\Delta\lambda$ . By these one can have the energy flux of all radiations within  $\Delta\lambda$  as.

$$\Phi(\lambda, T)\Delta\lambda$$

Thus:

"Any object that emits EM waves solely due to heating is called a black body radiator. We say that a black body radiator is an object in which matter and EM waves are in thermal equilibrium".

Here we need an energy distribution analysis based on all EM radiation spectrum (wavelength) as an *intensity distribution function*. Joseph Stefan and Boltzmann worked on this and their attempts led to Stefan-Boltzmann law to gives the total energy flux distributed from all wavelengths. Mathematically;

$$J_e = \sigma T^4$$

Where  $J_e$  energy flux or brightness is,  $\sigma$  is constant ( $5.67 \times 10^{-8}$ ) and T is absolute temperature of radiator in Kelvin. Thus we have a connection between energy flux and

temperature. This is because electromagnetic radiation which possesses pressure can be combined with the second law of thermodynamics (the law of entropy increase in a closed system).

This achievement along with Wien's displacement law and his formula for Kirchhoff function made Planck able to build his quantum theory of radiation. Wien's law stated that: " $\lambda_{\max}$  was inversely proportional to the temperature of the black body radiator". Mathematically:

$$\lambda_{\max} \propto T^{-1}$$

Plotting the curve of such a black body and reading off  $\lambda_{\max}$  from the curve, you can determine the temperature of the distant radiator for example a star. Wien expressed Kirchhoff function as:

$$\Phi(\lambda, T) = be^{-a/\lambda T} / \lambda^5$$

Where  $a$  and  $b$  are constants.

In this step Max Planck add the idea of entropy, to Wien's formula applied to high frequency and Rayleigh's formula applied to low frequency to express *spectral energy flux* in a more general way by his own special method of derivation in October 14 1900 as:

$$\Phi(\lambda, T) = \frac{b}{\lambda^5} \frac{1}{e^{a/\lambda T} - 1},$$

However Planck presented it in a modest title "An Improvement of Wien's Spectral Law" but this is the birth of *quantum theory*. In fact physics was just started again with his quantum theory or as Planck proposed "quantization of EM radiation". Formulation of statistical mechanics by Boltzmann's disclosed that number of particles in a thermodynamic substance is directly proportional to energy of particles i.e. increasing the energy of a particle lead to decreasing the number of such particles and vice versa but not come to zero due to existence of entropy. Consequently we have two fundamental results;

1. A black body radiates EM waves in the form of particles or bundles or quanta".
2. "The energy of each quantum is proportional to its frequency".

If,  $h$  be the constant of proportionality, the relation between energy of a quantum (particle) of EM radiation and its frequency ( $\lambda$ ) is:  $E = hc / \lambda$

This is called Planck relation and  $h$  is Planck constant equal to;  $6.625 \times 10^{-34}$  J/s.

The year 1905 was a special year in history of science and for Albert Einstein since in this year he exposed the special theory of relativity, disclosed the Brownian motion (microscopic random motion) and also described the

photoelectric (photon emissivity) effect. In photoelectric effect, "radiation of sufficiently high frequency/ short wavelength, imposing on certain substances, such as metals, causes emission of bound electrons with a maximum energy that varies linearly with the frequency of the radiation, i.e., to the entire energy of the photon". It is expressed as:

$$E_k = h\nu - \omega$$

$E_k$ : max KE of an emitted electron.  $h$ : the Planck constant.  
 $\nu$ : is the frequency of the radiation (of absorbed photon).  
 $\omega$ : is the energy necessary to remove the electron from the system.

It was the time that the concept of quanta brought back the wave-particle duality discussion again after 200 years from Newton. There was Law of Definite Proportions proposed in 1788 by the French chemist, Joseph Louis Proust (1754-1826), which states that "a chemical compound can contain elements in the ratio of whole numbers". In fact this was the birth of atomic theory. Although it was John Dalton (1766-1844) who empirically proved the existence of indivisible particles (Atoms) which every element in universe are made up from them. He noticed combination in different proportion gives different compounds. In fact these were

indivisible particles (atoms) tending to produce different compounds. Thus finally atom was discovered by Dalton. After him notions such as; atomic weight and mass or periodic table were introduced. It was also found that, the various atoms the various properties, thus there must exist even simpler entity causes this variations. The simplest atom was Hydrogen, thus they select it for searching those entities.

British physicist Joseph John Thomson (1856-1940), proposed in 1897 that: the so-called *cathode rays* produced in photoelectric effect were actually beam of negatively charged particles called electrons with an accurate charge-to-mass ratio ( $e/m$ ). Their mass was even smaller than lightest atom i.e. Hydrogen! Thus any atom must had a structure, and this electron must be only a part of it. Since an atom were known to be a neutral structure, thus we need an amount of positive charge as well, equal to the amount of electrons. But how they are distributed in an atom?

Thomson himself proposed a model with background positive charges along with the fix electrons around. But Ernest Rutherford (1871-1937) proved that this model is wrong, by some experiments using alpha particles. He and his coworkers at Cambridge bombarded the

background structure of the heavy atoms using alpha particles (such as electrons, neutrons, etc). They found no scattering of alpha particles on the background structure but with repeating the experiment they recognized backward scattering of alpha particle which were deflecting from the background structure. Rutherford proposed in 1911 that; this backward scattering of alpha particles is resulted from the collision of these particle with a more massive part of the atom. Apart with "a small intense positive charge of high inertia". He called it "positively charged core or *nucleus*" with about 1/100,000 the size of the atom itself which contains most of the mass of the atom along with orbiting electrons.

Using these, a Danish physicist Neil Bohr (885–1962), proposed a model similar to solar system, i.e. with the nucleus at the center and electrons orbiting around. Thus a planetary model so, the more distant electrons, the lesser acceleration, the lesser radiation and the closer distance, the more acceleration, the more radiation. This is because accelerating charges radiate EM waves by Planck-Einstein formula i.e.

$$E_{\text{rad}} = hf$$

This is continuous process thus in a very short radius the electron will collapse into the nucleus

in less than  $10^{-8}$  s. consequently this model also failed. However Bohr tried to improve his model using Planck-Einstein quantum theory to devise a stable model of the (hydrogen) atom. He proposed again that:

"The electron can be only on one of the infinitely many quantized orbits. The radius of the  $n$ th orbit is  $r_n = n^2 a_0$  where  $n$  is a positive integer. When the electron is in the  $n$ th orbit, its energy is  $E_n = -13.6/n^2$  eV. We also say that the atom is in its  $n$ th energy level.  $a_0 = 5.3 \times 10^{-11}m$ ".

In this new model electrons are not fixed and can make a transition from higher orbits to lower orbits. Thus we have difference of energy between orbits which carries with photons. Using emission spectral lines of gases of elements one can observe these photons since each element has its own unique set of spectral lines.

Although it was satisfactory at first but with more investigations a lot of question raised since in this model the angular momentum must be represented by an integer and the fate of electrons in first and higher orbits are not clarified.

French physicist Louis de Broglie (1892-1987) connected wave theory of electrons too particle theory by going back to the first principle of classical mechanics i.e. momentum. He searched

for a relation between wavelength and momentum. Again based on Planck-Einstein relation ( $E = hc/\lambda$ ), Einstein theory of relativity which connects the energy and momentum by

$$P = E/c$$

He proposed that:

"The momentum of a photon is related to its wavelength via  $p = h/\lambda$ ". Thus he made a Wave-particle duality basis for future research. Austrian physicist Erwin Schrödinger (1887-1961) was really fascinated by de Broglie's proposition of the wave nature of the electron (phase waves). With an extension and generalization he proposed his wave equation for the electron, using Planck constant ( $\hbar = h/2\pi$ ) similar to Bohr and De Broglie and introducing complex numbers into physics for the first time. He published his work in January 1926. His equation is;

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r}, t) \psi = i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t},$$

V: potential energy of the system, determined by the forces of interaction  $\Psi$  : Wave function.

Matrices were introduced into the mathematics at the end of the eighteenth century. In 1925 German physicist Werner Heisenberg (1901–

1976) developed matrix mechanics which also explained the spectral lines of the hydrogen atom. Thus with this new method an observable can assume a matrix from all possible values which it can have in a measurement. Multiplication of two matrix is not commutative thus they are incompatible. Knowing this Heisenberg exposed that if two physical observables are incompatible, their simultaneous measurement has an uncertainty. He proposed it is not possible to simultaneously determine the position ( $x$ ) and momentum ( $p$ ) of a particle i.e.

$$(\Delta x) \cdot (\Delta p) \geq \hbar,$$

Where  $\Delta$  represented uncertainty. This can also be stated for energy and time as:

$$(\Delta E) \cdot (\Delta t) \geq \hbar.$$

( $\hbar$ ) Says that only very small (sub-atomic) uncertainties (violations) in the measured quantities is allowed to be considered. Thus this principle have no use in our everyday scales.

Wait! What is  $\Psi(r, t)$  exactly? Heisenberg said it is electron waves. But we said that electrons are individual particles not waves. This question among some others led Max Born (1882-1970) to propose that  $\Psi(r, t)$  is a *probability amplitude*

which is a measurement of the quantum state (position, momentum, Energy etc.) of the particle would show it to be in the volume element  $dr$ . consequently;

$$\Psi(p) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int e^{\frac{-ip.r}{\hbar}} \Psi(r, t) dr$$

This gives us the exact quantum state of a particle in a given time. Because in "even the smallest macroscopic volume such as the tip of a sharp needle contains over a million trillion atoms, and the law of large numbers makes the probabilistic prediction of the quantum theory all but deterministic". Now we have Schrödinger-Born theory which can be applied to more complex atoms and molecules.

For this electron wave function to satisfy a wave equation based on the conservation of energy and momentum for the electron there are two methods; classical and a relativistic. The result is called *eigenvalue equations* which would solve by the methods of *eigenvalue problems* in mathematics.

$$E = mc^2$$

Where  $E$  = energy,  $m$  = mass and  $c$  = the speed of light in a vacuum. Einstein proposed this equation

as part of the special theory of relativity showing the relationship between mass and energy. In general and special theories of relativity we deal with mechanics of any system with large speeds near or at speed of light.

The basic structure in math and physics is **space**. German Mathematician Hermann Minkowski, (1864–1909) proposed the concept of space and time as a 4-dimensional entity. This concept is very significant for development of relativity theory.

Relativity theory fundamentally changed our views of the universe, and concepts of space and time. The special theory of relativity (STR) deals with the behavior of particles in different frames of reference which are moving at *constant relative speeds*. For these fast-travelling particles time passes more slowly. The general theory of relativity (GTR) proposed in 1916 describes *gravitational forces* in terms of the *curvature of space*, caused by the presence of mass. GTR deals with the behavior of particles in different frames of reference which are *accelerating relative* to one another.

**Chaos theory** begins with Laplace goes to Poincare' and in 1963, was in hand of meteorologist Edward Lorenz (1917–2008). He attempted to solve numerically the set of

differential equations of the simplest possible convective model of the atmosphere using early modern computers.

For more history one can study references for this part.

The note is that here we consider some core ideas along telling their story thus we will not consider them in elements.

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
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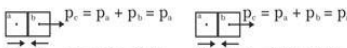
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## **Physics Elements**

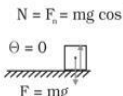
$$\begin{aligned}
 F_{12} &= -F_{21} & \Delta x &= x_f - x_i & d &= |\text{path}| & x_f &= x_i + v_{xi}t & a &= 0 \\
 \Sigma F &= ma & v_{x, \text{avg}} &= \frac{\Delta x}{\Delta t} & v_{\text{avg}} &= \frac{d}{\Delta t} & v_{\text{avg}} &= \frac{v_{xi} + v_{xf}}{2} & x_f &= x_i + \frac{1}{2}(v_{xi} + v_{xf})t \\
 p_x &= mv_x & v_x &= \frac{dx}{dt} & v &= |v_x| & x_f &= x_i + v_{xi}t + \frac{1}{2}a_x t^2 \\
 F_s &\leq \mu_s N & a_{x, \text{avg}} &= \frac{v_{xf} - v_{xi}}{t_f - t_i} & & & v_{xf} &= v_{xi} + a_x t & v_{xf} &= \frac{dx}{dt} \\
 F_k &= \mu_k N & a_x &= \frac{dv_x}{dt} = \frac{d^2x}{dt^2} & & & v_{xf}^2 &= v_{xi}^2 + 2a_x(x_f - x_i) \\
 & & & & & & a_{xf} &= a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}
 \end{aligned}$$
  


$$F = ma$$
  


$$p_x = m_x v_{xi}$$
  


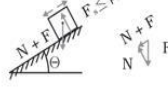
$$p_c = p_a + p_b = p_a$$

$$p_c = (m_x + m_y)v_{xi}$$

$$0_{xi} \quad 0_{xi} \quad p_c = (m_x + m_y)v_{xi}$$
  


$$N = F_s = mg \cos \theta$$

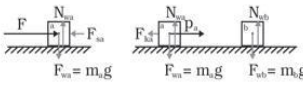
$$\theta = 0$$

$$F = mg$$
  


$$F \leq \mu_s N$$

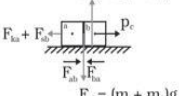
$$N + F$$

$$N$$

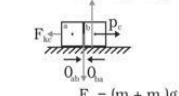
$$F$$
  


$$F_{ax} = m_x g$$

$$F_{bx} = m_x g$$

$$F_{cx} = m_x g$$
  


$$F_{ax} + F_{bx} = F_c$$

$$F_{ax} + F_{bx} = (m_x + m_y)g$$
  


$$F_{ax} + F_{bx} = (m_x + m_y)g$$

"The quantitative laws of physics are expressed by *mathematical equations*, whose symbols are directly or indirectly related to *the numerical results of measurements*".

In practice, physics starts with measuring of a natural phenomenon. The measurement will be represented in numerical symbols i.e. integers, real numbers, complex numbers etc. Nature is describable in terms of mathematical equations and functions or a set of them which are our models of reality. Although we mustn't ignore the need for *philosophy of research* and *logic* to build our laws in a rigorous scientific approach. These are because philosophy is the first window to

universe, the next window is mathematical logic for a particular problem in a particular space (mathematical or physical). The final window is a suitable mathematical tool to get our model of reality. We will start with units and then represent fundamental laws in a simple way.

We can describe nature in terms of four fundamental quantities: *length*, *mass*, *time*, and *electric charge* in a metric system of units. All other quantities can be expressed in terms of these four. Smaller or larger lengths can be expressed as multiples of powers of 10. To clarify see some samples:  $0.0000000005 \text{ m} = 5.0 \times 10^{-10} \text{ m}$ ; is the size of the hydrogen atom.

$300000000 \text{ m/s} = 3.0 \times 10^8 \text{ m/s}$ ; is the speed of light in vacuum.

The form  $10^n$  conveys no significant figures. Power of ten, makes a precision and represent very large or very small numbers in a compact way.

$$0.000\ 009\ 8 = 9.8 \times 10^{-6}$$

A standard place for the decimal point is usually after the first significant figure. We can do simple arithmetics with them following the rule of powers.

$$(a \times 10^m) \times (b \times 10^n) = (ab) \times 10^{(n+m)}$$

$$(6 \times 10^4) \times (3 \times 10^6) = 18 \times 10^{10} = 1.8 \times 10^{11}.$$

$$(6 \times 10^4) \div (3 \times 10^{-6}) = 18 \times 10^{-2} = 1.8 \times 10^{-1},$$

$$(a \times 10^m) \div (b \times 10^n) = (a/b) \times 10^{(m-n)}$$

$$(6 \times 10^4) \div (3 \times 10^6) = 2 \times 10^{-2}$$

$$(6 \times 10^4) \div (3 \times 10^{-6}) = 2 \times 10^{10}.$$

We also have multipliers of power of ten.

Prefix	Symbol = (multiplying factor)
deca <sup>a</sup>	da = 10
hecto <sup>a</sup>	h = 10 <sup>2</sup>
kilo	k = 10 <sup>3</sup>
mega	M = 10 <sup>6</sup>
giga	G = 10 <sup>9</sup>
tera	T = 10 <sup>12</sup>
peta <sup>a</sup>	P = 10 <sup>15</sup>
exa <sup>a</sup>	E = 10 <sup>18</sup>

deci <sup>b</sup>	$d = 10^{-1}$
centi <sup>c</sup>	$c = 10^{-2}$
milli	$m = 10^{-3}$
micro <sup>d</sup>	$\mu = 10^{-6}$
nano	$n = 10^{-9}$
pico	$p = 10^{-12}$
femto	$f = 10^{-15}$
atto	$a = 10^{-18}$

However, according to "The Cambridge Handbook of Physics Formulas", International System of units (SI) uses seven base units as bellow:

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>
length	metre <sup>a</sup>	m
mass	kilogram	kg
time interval	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

<sup>a</sup>Or "meter".

All other units are derived from these seven basic units.

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>	<i>equivalent</i>
catalytic activity	katal	kat	$\text{mol s}^{-1}$
electric capacitance	farad	F	$\text{C V}^{-1}$
electric charge	coulomb	C	A s
electric conductance	siemens	S	$\Omega^{-1}$
electric potential difference	volt	V	$\text{J C}^{-1}$
electric resistance	ohm	$\Omega$	$\text{V A}^{-1}$
energy, work, heat	joule	J	N m
force	newton	N	$\text{m kg s}^{-2}$
frequency	hertz	Hz	$\text{s}^{-1}$
illuminance	lux	lx	$\text{cd sr m}^{-2}$
inductance	henry	H	$\text{V A}^{-1} \text{s}$
luminous flux	lumen	lm	cd sr
magnetic flux	weber	Wb	V s
magnetic flux density	tesla	T	$\text{V s m}^{-2}$
plane angle	radian	rad	$\text{m m}^{-1}$
power, radiant flux	watt	W	$\text{J s}^{-1}$
pressure, stress	pascal	Pa	$\text{N m}^{-2}$
radiation absorbed dose	gray	Gy	$\text{J kg}^{-1}$
radiation dose equivalent <sup>a</sup>	sievert	Sv	$[\text{J kg}^{-1}]$
radioactive activity	becquerel	Bq	$\text{s}^{-1}$
solid angle	steradian	sr	$\text{m}^2 \text{m}^{-2}$
temperature <sup>b</sup>	degree Celsius	$^{\circ}\text{C}$	K

<sup>a</sup>To distinguish it from the gray, units of  $\text{J kg}^{-1}$  should not be used for the sievert in practice.

<sup>b</sup>The Celsius temperature,  $T_{\text{C}}$ , is defined from the temperature in kelvin,  $T_{\text{K}}$ , by  $T_{\text{C}} = T_{\text{K}} - 273.15$ .

Another important thing to mention here is the dimensions that these units act. The dimensional basis are length (L), mass (M), time

(T), electric current (I), temperature ( $\Theta$ ), amount of substance (N), and luminous intensity (J).

acceleration	$a$	$L T^{-2}$	$m s^{-2}$
action	$S$	$L^2 M T^{-1}$	$J s$
angular momentum	$L, J$	$L^2 M T^{-1}$	$m^2 kg s^{-1}$
angular speed	$\omega$	$T^{-1}$	$rad s^{-1}$
area	$A, S$	$L^2$	$m^2$

And so on and so forth. For instance in angular momentum; the dimensions are  $L^2MT^{-1}$  since the unit is  $m^2kgs^{-1}$ .

Greek alphabets are;

$A$	$\alpha$	alpha	$N$	$\nu$	nu
$B$	$\beta$	beta	$\Xi$	$\xi$	xi
$\Gamma$	$\gamma$	gamma	$O$	$o$	omicron
$\Delta$	$\delta$	delta	$\Pi$	$\pi$	pi
$E$	$\epsilon$	$\varepsilon$ epsilon	$P$	$\rho$	$\varrho$ rho
$Z$	$\zeta$	zeta	$\Sigma$	$\sigma$	$\varsigma$ sigma
$H$	$\eta$	eta	$T$	$\tau$	tau
$\Theta$	$\theta$	$\vartheta$ theta	$\Upsilon$	$\upsilon$	upsilon
$I$	$\iota$	iota	$\Phi$	$\phi$	$\varphi$ phi
$K$	$\kappa$	kappa	$X$	$\chi$	chi
$\Lambda$	$\lambda$	lambda	$\Psi$	$\psi$	psi
$M$	$\mu$	mu	$\Omega$	$\omega$	omega

We see constants in formulas and unite conversions in problems. Below we have some basic and more common ones.

Base Units	Symbol	Unit
Mass of electron	$m_e$	$9.11 \times 10^{-31} \text{ kg}$
Mass of proton	$m_p$	$1.67 \times 10^{-27} \text{ kg}$
Avogadro Constant	$N_A$	$6.022 \times 10^{23} \text{ mol}^{-1}$
Elementary charge	$e$	$1.602 \times 10^{-19} \text{ C}$
Faraday Constant	$F$	$96,485 \text{ C/mol}$
Speed of light	$c$	$3 \times 10^8 \text{ m s}^{-1}$
Molar Gas Constant	$R$	$8.314 \text{ J mol}^{-1} \text{ K}^{-1}$
Boltzmann Constant	$k$	$1.38 \times 10^{-23} \text{ J K}^{-1}$
Gravitation Constant	$G$	$6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Permeability of Space	$\mu_0$	$4\pi \times 10^{-7} \text{ N/A}^2$
Permittivity of Space	$\epsilon_0$	$8.85 \times 10^{-12} \text{ F/m}$

Angle	° (degree)	$180^\circ = \pi \text{ rad}$
Volume	Liter	$1 \text{ L} = 1 \text{ dm}^3$
Energy	Erg	CGS unit ( $\text{g cm}^2/\text{s}^2$ ) $1 \text{ erg} = 10^{-7} \text{ J}$
ΕΥΓΕΙΣ	ΛΟΓΙ ΕΙΣΕΓΓΕΙΟΝ	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$
Force	Dyne	CGS unit ( $\text{g cm/s}^2 = \text{erg/cm}$ ) $1 \text{ dyne} = 10^{-5} \text{ N}$
Pressure	Bar	$1 \text{ Bar} = 10^5 \text{ Pa}$
Length	Angstrom	$1 \text{ \AA} = 1 \times 10^{-10} \text{ m}$

We represent our elements in a way that we did in mathematics. Since we seek for the basic blocks that built this beautiful building.

Motion is the most basic block of physics since we live in space and time or space-time as Einstein relativity suggested. Every particle and object move in this area. In classical way of motion, when we say it moves we talk about kinematics and when we say why it moves we talk about dynamics. Thus we begin with the first most generalized formulation of motion by Newton.

Newtonian mechanics says; given the present, predict the future. To predict the future trajectory of a particle from its present condition we basically use two main aspects of motion, i.e. *velocity* and *acceleration*. This practices from uniform motions and freely falling objects to circular and oscillatory ones.

A *uniform motion* happens when the trajectory of a particle is a straight line i.e. the distance covered by  $P(x-x_0)$  is proportional to the time passed.

$$x - x_0 = v_x t$$

Where  $v_x$  is the constant of proportionality and the velocity of  $P$  in the  $x$  direction; the coordinate  $x_0$  is called the initial value of  $x$ .

The distance is a value that being positive or negative does not matter for it. Thus:

$$l = |x - x_0|$$

Consequently,  $v$  as speed of the particle is also an absolute value.

$$v = |v_x|$$

One can say that:  **$l = vt$  or  $v = l/t$**

Which are more common notations. So, Speed (velocity) is distance traveled per unit time (meter/sec).

From basic algebra we know that, the slope of an equation is a ratio of two distances covered in a coordinate system. Here this slope is called *velocity* and is the *first derivative* (rate of changes) of position since it says that how much of position is covered with respect to the time passed represented by a straight line. i.e

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt},$$

Which is also called *time derivative*. If the velocity changes in the path of time, then the motion is called *accelerated* represented by a curve. i.e.

$$a_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt},$$

Which is called *second derivative* of position.

This is represented as:

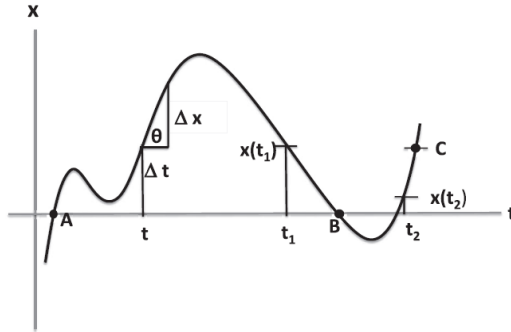
$$a_x = \frac{d(\frac{dx}{dt})}{dt} = a_x = \frac{d^2x}{dt^2}$$

According to these the unit of acceleration in SI is:

$$\frac{1(\frac{meter}{sec})}{1 sec} = 1 \frac{meter}{sec^2}$$

Constancy it is equivalent to uniformity. Constant velocity motion means no dependence existed and constant acceleration motion means no change in velocity in time interval such as in

freely falling objects. If one want to show our discussion by a graph, it would be like this.



$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Average velocity

$$\bar{a} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}.$$

Average acceleration

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

Instantaneous velocity

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Instantaneous acceleration

The most general expression for the position of a particle with constant (uniform) acceleration  $a$  is

$$x(t) = \frac{1}{2}at^2 + bt + c$$

If this being considered for a particle in  $-g$  ( $9.8 \text{ m/s}^2$ ), the earth uniform gravitational acceleration then:

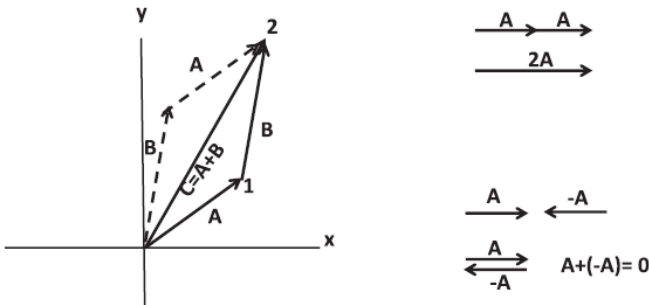
$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0.$$

Where  $v_0$  and  $y_0$  are instead of  $b$  and  $c$  as initial velocity and position for trajectory  $y(t)$  respectively. For trajectory  $x(t)$  we have:

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0.$$

To continue we need vector calculus or vector analysis. We know that, a point in space can be represented by its ordered pairs  $(a_1b_1c_1 \dots n_1)$  which means, to have this point we moved from a point such as origin and reached to this point with this coordinate. Thus it is clear that we moved simultaneously in every path related to every pair. This path is called a *vector* since it represent both *magnitude* (amount, distance etc.) and *direction* of that pair. Therefore we see that, every path in space can be represented by a vector

quantity. This quantity is composed of some parts to form it which is called *components* of a vector. "The magnitude is how long it is, and direction is its angle relative to some fixed direction, usually the  $x$ -axis". We can do all four simple arithmetic operations here, but with some simple rules. Opposite to vector quantity we have *scalar* quantity with no direction and only magnitude. One can also multiply a scalar with a vector to promote that vector. If you subtract a vector from itself you get a vector with zero length called, zero or *null vector*. The negative (minus) vector is the same vector pointing the opposite way. Pictorially;



From a general point of view we need to have special vectors. They are called *unit vectors*. Because they have unit length. We tag them as  $i$ ,  $j$  and  $k$

pointing along the  $x$ ,  $y$ , and  $z$  axes. If we have some vector  $\mathbf{A}$ , it defines as:

$$\mathbf{A} = iA_x + jA_y + kA_z$$

Which is is the vector sum of  $\mathbf{A}$  as well. Its magnitude (absolute value) is;

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

If the vector  $\vec{A}$  in 3-D space is represented with an initial point at the origin,  $O = (0, 0, 0)$  and its end point at some point,  $P = (A_x, A_y, A_z)$  then,  $iA_x$ ,  $jA_y$ ,  $kA_z$  are *component vectors* in  $x$ ,  $y$ ,  $z$  directions of  $\vec{A}$  and  $A_x, A_y, A_z$  are components of  $\vec{A}$ .

In a more general speaking, If  $P(x, y, z)$  is a point in 3-D space, then we can define a vector  $\mathbf{r}$  from the origin  $\mathbf{O}$  to the point  $\mathbf{P}$  and called it the *position vector* (or *radius vector*).

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

With magnitude;

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

If we have two vectors as;  $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$   
 and  $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ . in x, y, z  
 directions which is represented as 1, 2, 3 here,  
 then we have:

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}$$

$$m\mathbf{A} = m(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) = (mA_1)\mathbf{i} + (mA_2)\mathbf{j} + (mA_3)\mathbf{k}$$

\*corresponding components are added and multiplied.

The second property in its more general form  
 gives a *linear combination* of vectors.

If  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$  and  $a_1, a_2, \dots, a_n$  are  
 sequences of vectors and scalars respectively,  
 according to second property we have a vector as:

$$\mathbf{B} = a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + \dots + a_n\mathbf{A}_n$$

The vector  $\mathbf{B}$  is called a *linear combination* of  
 the vectors  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ . therefore we define;

Vectors  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$  are *linearly dependent* if  
 there exist scalars  $a_1, a_2, \dots, a_n$ , not all zero, such  
 that:

$$a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + \dots + a_n\mathbf{A}_n = \mathbf{0}$$

Otherwise, the vectors are *linearly independent*.  
 Thus we can have the vector equation

$$x_1 \mathbf{A}_1 + x_2 \mathbf{A}_2 + \cdots + x_n \mathbf{A}_n = \mathbf{0}$$

$x_1, x_2, \dots, x_n$  are equivalent of  $a_1, a_2, \dots, a_n$  and unknowns. Consequently:

$$x_1 = 0, x_2 = 0, x_3 = 0, \dots, x_n = 0$$

Always hold. By these we say:

Two or more vectors are *linearly dependent* iff one of them is a linear combination of the others which means one is a multiple of the other. And remember:

$$m\mathbf{A} = \mathbf{0} \quad \text{and} \quad \mathbf{A} \neq \mathbf{0}, \text{ implies } m = 0$$

According to discussion above one can define *vector field* and *scalar field* to have a *scalar function of position*  $\phi(x, y, z)$  i.e. a point and *vector function of position*  $\mathbf{V}(x, y, z)$  i.e. a vector, in a region of space.

The *dot product* of magnitude of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is a scalar:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos \theta, \quad 0 \leq \theta \leq \pi \quad \text{With}$$

properties similar two real numbers.

The *cross product* of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is a vector which is perpendicular to the plane of  $\mathbf{A}$

and  $\mathbf{B}$ . where  $\mathbf{u}$  is a unit vector indicating the direction of  $\mathbf{A} \times \mathbf{B}$ .

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \theta \mathbf{u} \quad 0 \leq \theta \leq \pi$$

If  $\mathbf{A} = \mathbf{B}$ , or if  $\mathbf{A} \parallel \mathbf{B}$  ( $\mathbf{A}$  parallel to  $\mathbf{B}$ ), then  $\sin \theta = 0$  which implies  $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ .

Given

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \text{ and } \mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}.$$

We have:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \\ & \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \mathbf{k} \end{aligned}$$

Multiplying three vectors is called *triple products* with the form of;

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C}, \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}), \text{ and } \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

If  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are vectors and  $m$  is a scalar then:

In general,  $(\mathbf{A} \cdot \mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$ .

In general,  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

(Associative Law for Cross Products Fails)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Given

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

$$\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}, \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}.$$

Then;

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

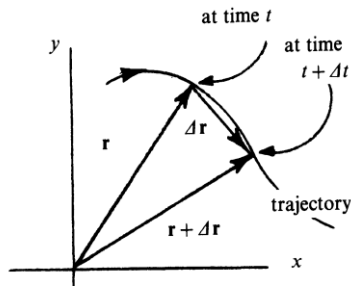
The sets  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and  $\mathbf{a}', \mathbf{b}', \mathbf{c}'$  are reciprocal sets of vectors iff;

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}$$

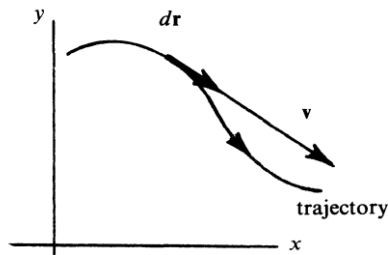
We defined the position vector as:  
 $r = xi + yj + zk$  and also its derivative as:

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t}$$

Here  $\Delta \mathbf{r}$  is the displacement vector corresponding to a change in the particle's position during  $\Delta t$  as bellow.



Since  $\mathbf{v} = d\mathbf{r}/dt$ ,  $\mathbf{v}$  has the direction of  $d\mathbf{r}$ . Therefore it points *tangentially* forward along the trajectory.



In fact

$$\mathbf{v} = |\mathbf{v}| = \frac{dr}{dt}$$

Differentiating a vector means differentiating its components. For vector valued function of velocity  $\mathbf{v}(\mathbf{t})$  we have;

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

In general If  $\mathbf{u}$  be a single scalar variable then:

$$\frac{d\mathbf{r}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

For acceleration;

$$\mathbf{a} = \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

A function that is differentiable is essentially continuous but the opposite is not true. This is right for vector-valued function as well.

Suppose  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are differentiable vector functions of a scalar  $\mathbf{u}$ , and  $\phi$  is a differentiable scalar function of  $\mathbf{u}$  then:

$$\begin{aligned}
\frac{d}{du}(\mathbf{A} + \mathbf{B}) &= \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \\
\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \cdot \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B} \\
\frac{d}{du}(\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \times \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \times \mathbf{B} \\
\frac{d}{du}(\phi \mathbf{A}) &= \phi \frac{d\mathbf{A}}{du} + \frac{d\phi}{du} \mathbf{A}
\end{aligned}$$

And also;

$$\begin{aligned}
\frac{d}{du}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) &= \\
\mathbf{A} \cdot \mathbf{B} \times \frac{d\mathbf{C}}{du} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \times \mathbf{C} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B} \times \mathbf{C} \\
\frac{d}{du}\{\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\} &= \\
\mathbf{A} \times \left(\mathbf{B} \times \frac{d\mathbf{C}}{du}\right) + \mathbf{A} \times \left(\frac{d\mathbf{B}}{du} \times \mathbf{C}\right) + \frac{d\mathbf{A}}{du} \times (\mathbf{B} \times \mathbf{C})
\end{aligned}$$

Suppose that  $\mathbf{A}$  be a vector depending on more than one variable, say  $x, y, z$ , i.e.

$$\mathbf{A} = \mathbf{A}(x, y, z).$$

If the limit of  $\mathbf{A}$  exists then the **partial derivative** of  $\mathbf{A}$  with respect to  $x, y, z$  would be:

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\mathbf{A}(x + \Delta x, y, z) - \mathbf{A}(x, y, z)}{\Delta x}$$

$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{\mathbf{A}(x, y + \Delta y, z) - \mathbf{A}(x, y, z)}{\Delta y}$$

$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \rightarrow 0} \frac{\mathbf{A}(x, y, z + \Delta z) - \mathbf{A}(x, y, z)}{\Delta z}$$

Higher derivative can also be defined as:

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial \mathbf{A}}{\partial x} \right), \quad \frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial \mathbf{A}}{\partial y} \right), \quad \frac{\partial^2 \mathbf{A}}{\partial z^2} = \frac{\partial}{\partial z} \left( \frac{\partial \mathbf{A}}{\partial z} \right)$$

$$\frac{\partial^2 \mathbf{A}}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial \mathbf{A}}{\partial y} \right), \quad \frac{\partial^2 \mathbf{A}}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \mathbf{A}}{\partial x} \right), \quad \frac{\partial^3 \mathbf{A}}{\partial x \partial z^2} = \frac{\partial}{\partial x} \left( \frac{\partial^2 \mathbf{A}}{\partial z^2} \right)$$

If  $\mathbf{A}$  and  $\mathbf{B}$  are vector functions of  $x, y, z$ . Then:

$$\frac{\partial}{\partial x} (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}$$

$$\frac{\partial}{\partial x} (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B}$$

$$\begin{aligned} \frac{\partial^2}{\partial y \partial x} (\mathbf{A} \cdot \mathbf{B}) &= \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial x} (\mathbf{A} \cdot \mathbf{B}) \right\} = \frac{\partial}{\partial y} \left\{ \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B} \right\} \\ &= \mathbf{A} \cdot \frac{\partial^2 \mathbf{B}}{\partial y \partial x} + \frac{\partial \mathbf{A}}{\partial y} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y \partial x} \cdot \mathbf{B}, \end{aligned}$$

And so on and so forth.

If  $\mathbf{A}$  and  $\mathbf{B}$  are functions of  $x, y, z$ . Then:

$$\text{If } \mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

$$d\mathbf{A} = dA_1\mathbf{i} + dA_2\mathbf{j} + dA_3\mathbf{k}$$

$$d(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot d\mathbf{B} + d\mathbf{A} \cdot \mathbf{B}$$

$$d(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times d\mathbf{B} + d\mathbf{A} \times \mathbf{B}$$

$$\text{If } \mathbf{A} = \mathbf{A}(x, y, z),$$

Then;

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x} dx + \frac{\partial \mathbf{A}}{\partial y} dy + \frac{\partial \mathbf{A}}{\partial z} dz$$

And so on and so forth. If one use differential calculus in the study of geometry of curves and surfaces, then this is called *differential geometry*. Mechanics is often the study of the motion of particles along curves i.e. *kinematics* which is about how a particle move. If one consider why a particle move he/she deals with forces which is called *dynamics*. Newton's law are fundamental in this respect.

If  $\mathbf{F}$  is the *net force* acting on an object of *mass*  $\mathbf{m}$  moving with velocity  $\mathbf{v}$ , we have:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$$

Where  $\mathbf{mv}$  is the momentum of the object. If  $\mathbf{m}$  is constant, then

$$\mathbf{F} = \mathbf{m} (d\mathbf{v}/dt) = \mathbf{ma},$$

Where  $\mathbf{a}$  is the acceleration of the object. Again we notice that;

$$m \vec{a} = \vec{F},$$

This can also be written as:

$$m \frac{d^2 x(t)}{dt^2} = F_x \quad \text{or} \quad m\ddot{x} = F_x$$

Do not forget that;

"A greater mass  $m$  goes with a smaller acceleration  $|a|$ . Thus, mass constitutes **inertia**; that is, it tends to oppose attempts at acceleration."

In terms of the momentum ( $\mathbf{p} = \mathbf{mv}$ );

$$P_x = m\dot{x}$$

Thus we can write;

$$\dot{p}_x = F_x.$$

We know that, the motion of a particle must be referred to a frame of reference (a coordinate system) to consider its momentum, velocity and acceleration. Now we say that it also possible in

the existence of forces no motion occurs at all, this is called an *equilibrium state*. This happens when the existing forces cancel each other. From Newton's first law (law of inertia) we know that "an isolated particle maintains a uniform motion". Consequently, a frame in which a particle experiences such a motion is called an *inertial frame of reference*. Almost "all measurements in physics are assumed to be performed with respect to an inertial system".

In our familiar mathematical point of view, one can summarize all we said as follows.

In Newtonian mechanics, the dynamics of a system of  $N$  particles satisfy their coordinate trajectories as a function of time through the usual vector-based coordinates  $r_i(t)$  for  $i \in \{1, 2, \dots, N\}$ . However, after Newton, people said that it can be done by generalized coordinates  $q_i(t)$  for  $i \in \{1, 2, \dots, 3N\}$  in a 3-D space. These *generalized coordinates* could be vectors, angles, etc. Likewise, we have:

$$v_i \equiv \dot{r}_i \quad \text{or} \quad v_i \equiv \dot{q}_i$$

$$P_i = m_i v_i \quad \text{or} \quad P_i = p_i$$

For spatial and generalized coordinates respectively.

If we have a fixed set of masses  $m_i$ , then  $F = ma$  can be expressed as:

$$F_i = \frac{d(m_i \dot{r}_i)}{dt} \quad \text{or} \quad F_i = \dot{p}_i$$

**Energy** is a more abstract and practical concept than motion, mass and force. In general, mechanics is dealing with two forms of energy; *kinetic energy* and *potential energy* such as  $E_m = K_E + P_E$ . In the case of energy transfer such as, heat or noise, there may be a loss of energy which is called *dissipation*. Its dimensions and SI unit are  $ML^2T^{-2}$  and joule respectively.

Potential energy is the energy depends on the particle position. Conservative forces can be derived from a potential  $V(x)$  as;

$$F_x = -\frac{dV(x)}{dx},$$

It says that; alongside a trajectory  $x(t)$ , the potential  $V(x(t))$  fluctuates at a rate  $\dot{V} = \frac{dV}{dx} \dot{x}$  which implies that;

$$\dot{x}m\ddot{x} - \dot{x}F_x = \frac{dE}{dt}$$

Thus energy is conserved and then;

$$E = T + V$$

One can write this as:

$$E_{tot} = K_E + P_E$$

i.e.; fixed total energy of a system is equal to a sum of kinetic energy which depends on velocity and potential energy which depends on coordinate (position) of the particle vary in dish of time.

The kinetic energy is defined as:

$$K_E = \frac{1}{2}mv^2 \quad \text{or} \quad T = \frac{m}{2}\dot{x}^2$$

Consequently;

$$\Delta E = \Delta T + \Delta V$$

By this infinitesimal increment definition we can say:

$$E = \sum_i \frac{1}{2}m_i v_i^2 + U(r_1, r_2, r_{13}, \dots)$$

Where U is another common symbol for potential energy.

"Thus the energy of the system can be written as the sum of two quite different terms: the kinetic energy, which depends on the velocities, and the potential energy, which depends only on the co-ordinates of the particles".

According to Newton's Second Law;

$$m(dv/dt) = F,$$

Consequently;

$$dT/dt = Fv.$$

If momentum be "the force in the direction of motion" then:

$$\frac{dT}{dt} = \mathbf{mv} \frac{dv}{dt} = \mathbf{F}_t \mathbf{v}$$

Here  $F_t$  is the *tangential force* i.e. the force tangent to the motion of the particle. The work

done by a force  $F$  during the time interval from  $t_1$  to  $t_2$  is equal to:

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \, dt \quad \text{or} \quad \int_{t_1}^{t_2} Fv(t) \, dt$$

Thus  $W = F \times S$  where  $S$  is the distance. The *momentum vector* of a particle is defined as,  $p = mv$ . For a system of many particle we have the *total momentum vector* as:

$$\mathbf{p}_{\text{syst}} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + \cdots = \sum_i m_i \mathbf{v}_i$$

We know that; "The total force on a particle equals the rate of change of its momentum". We also know that in the absence of external forces the total momentum of a system of two particle is conserved due to the newton third law i.e.  $F_1 + F_2 = 0$ . This is also hold for a system of many particle such as;

"The total momentum vector of an isolated system is conserved" i.e.

$$\begin{aligned} \frac{d\mathbf{p}_{\text{syst}}}{dt} &= \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_3}{dt} + \cdots \\ &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = 0 \end{aligned}$$

For a non-isolated system of particles, interacting with each other, after reactions, the external forces would left over which is called the *net force*.

If we have a collision with conserved or unchanged kinetic energy then we say it's an

*elastic* collision. This can more happen among, atoms, particles, molecules due to the lack or absence of friction. "The relative velocity of two particles is the same before and after they collide elastically". Thus "the force is proportional to the displacement". It is also called Hooke's law deal with small distortions of particles with restored condition after collision.

There exist a point whose motion is the best reprehensive of a whole system called the *center of mass* in which the total mass of the system is concentrated at that point. Symbolically;

$$\mathbf{r}_c = \frac{\sum_i m_i \mathbf{r}_i}{m_{\text{syst}}}$$

Where  $\mathbf{r}$  is the position vector. For continuous system we have:

$$\mathbf{r}_c = \frac{\int \mathbf{r} dm}{m_{\text{syst}}}$$

A circle is a curve. A particle with a circular motion or orbit essentially experiences forces and accelerations like in linear motions. We apply laws of circular motion to rigid bodies such as planets, satellites etc. and also to theoretical particles. Instead we have:

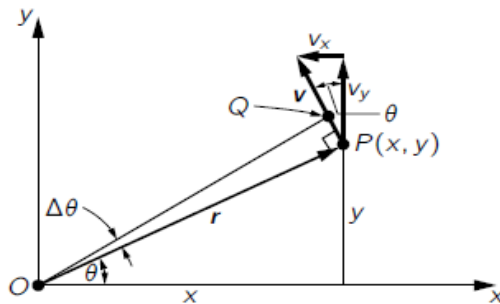
Angular position  $\theta$ , the angular velocity  $\omega$ , and the angular acceleration  $\alpha$ .

The simplest is a *rigid body* motion which an extended body rotates about a *fixed axis*. A rigid body is a system of particles whose mutual distances are constant. Thus any point on such a body moves in a plane perpendicular to this axis. Therefore we have *Rotation* in a plane or in 2-D space.

In comparison with linear motion here we deal with *angular change of the position* of the whole object, from one time to another. Thus;

$$\frac{d\theta}{dt} = \omega, \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Thus the orbit can be used as a **coordinate** in order to specify the position, velocity, etc. of a particle or body in revolution.



Remember that  $\theta$  is proportional to the ratio between the arc  $s$  and the radius  $r$  of the circle.

$$\theta = \frac{s}{r}, \quad s = r\theta$$

The unit is radian as;

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ \quad \frac{360^\circ}{1 \text{ rad}} = 2\pi$$

The frequency,  $f$ , is defined as;

$$\frac{\text{number of cycles}}{\Delta t}$$

In a uniform circular motion;

$$f = \frac{1}{T} \quad , \quad \omega = 2\pi f \text{ or } \omega = \frac{2\pi}{T}$$

$T$ , is the *period* which is the time required for one revolution and  $\omega$  is the angular frequency. Also the linear velocity vector ( $v = \frac{ds}{dt}$ ) becomes, angular velocity vector ( $v = \frac{dr}{dt}$ ) with the magnitude of  $v = \omega r$  and direction at  $90^\circ$  to the left or right of  $r$  in a 2-D plane of the motion. Both  $r$  and  $v$  are phase vectors since they have constant magnitude and angular velocity. The successive time derivatives of the phase vector ( $\frac{dr}{dt}$ ) with magnitude  $|\omega r|$  rotate  $r$  by successive  $90^\circ$ . If set  $\frac{dr}{dt} = v$  then,  $\frac{dv}{dt} = a$ , is the acceleration which is a phase vector of magnitude  $|\omega v|$  rotate  $v$  by successive  $90^\circ$ . Thus we write  $v = \omega r$  and then;

$$a = \omega^2 r = \frac{v^2}{r}$$

For a particle with a mass  $m$ , according to Newton's second law we have:

$$F_{\text{tot}} = m\omega^2 r = \frac{mv^2}{r}$$

If the force, velocity and acceleration are toward the center of the orbit; they are *centripetal* i.e. "orthogonal to the motion itself" if not they are *centrifugal*.

In a complete **revolution** (an arc  $\pm 2\pi r$ ) the original position of the particle would be restored. Thus, any position can be specified by infinitely many alternative coordinates.

$$s, \quad s + 2\pi r, \quad s - 2\pi r, \quad s + 4\pi r$$

The velocity,  $v$  is:

$$v = \left| \frac{ds}{dt} \right|$$

To identify whether the motion is counterclockwise or clockwise;

$$v_\theta = \frac{ds}{dt} = \pm v = |v_\theta|$$

Where plus for counterclockwise and minus for clockwise motion.  $v_\theta$  is called tangential velocity. Since  $\frac{d\theta}{dt} = \omega = \frac{v_\theta}{r}$  and  $s = r\theta$  then;

$$v_\theta = r\omega$$

is the magnitude of the velocity. In a more clear way one can write.

$$r\Delta\theta/\Delta t$$

Non-uniform circular motion gives a non-pure centripetal acceleration vector  $\mathbf{a}$  with a tangential component  $\mathbf{a}_\theta$  and a radial component  $\mathbf{a}_r$ . Where;

$$|\mathbf{a}_r| = \omega^2 r = \frac{v^2}{r} \quad \mathbf{a}_\theta = \frac{dv}{dt}$$

To having a linear motion there must exist a force, and to make something rotate we need to have a rotary/twisting force, which is called a *torque*. For defining a force we say how much *work* it does when it acts through a given displacement. For defining a torque we say how much *work* is done in turning an object. We do this by keeping "analogy between linear and angular quantities". We know that;

$$W = F \times S$$

Where W is work, F is force and S is distance. Now we say that;

$$W = \tau \times \theta$$

Where  $\tau$  is the torque and  $\theta$  is the angle.

Now imagine a rotating rigid body with various forces acting on it, applied at a certain point (x, y), then

$$\Delta W = F_x \Delta x + F_y \Delta y.$$

In linear motion becomes;

$$\Delta W = (xF_y - yF_x)\Delta\theta.$$

Thus the change in work is equal to the torque times the change in angle. It is clear that, torque must have a definition in terms of the force. In the case of acting several forces we have a sum of works, each depends on  $\Delta\theta$ .

$$\tau_i = x_i F_{yi} - y_i F_{xi}$$

For an object in equilibrium;

$$\Delta w = \tau \Delta\theta = 0$$

Thus;  $\sum \tau_i = 0$

"The amount of twist, or torque, is proportional both to the radial distance and to the tangential component of the force". The torque can also be the magnitude of the force times the length of the lever arm.

As  $F_{EXT} = \frac{dp}{dt}$  and  $p$  is the total momentum of a system of particles, then;

$$\tau_{EXT} = \frac{dL}{dt}$$

Where  $L$  is the *angular momentum* i.e. "an amount of spin". What diminishes the effect of torque is called *moment of inertia* ( $I$ ) which is analogues of mass. It gives kinetic energy to a spinning object in the form of;

$$k = \frac{1}{2} I \omega^2$$

And the total torque will be defined as;

$$\tau_{total} = I\alpha$$

Instead of;

$$k = \frac{1}{2} m v^2 \text{ and } F_{total} = m a$$

*Moment of inertia* can be determined by shape, the mass and the axis of rotation in a rigid body motion i.e. the structure of an object. For a single particle we have;

$k = \frac{1}{2} (m r^2) \omega^2$  which implies that  $I = m r^2$  for a single particle. For a rigid system of many-particle, we need to obtain the total moment of inertia. Similar to the case of single particle we have;

$$k = \frac{1}{2} I_1 \omega^2 + \frac{1}{2} I_{21} \omega^2 + \dots$$

If  $m_i$  be the masses of the particle of the rigid body system and  $r_i$  be the distances of the particles to the common axis of rotation then;

$$I = m_1 r_1^2 + m_2 r_2^2 + \dots$$

$$= \sum_i m_i r_i^2$$

For a continuous system of infinite particles with infinitesimal mass ( $dm$ ) we have:  $dI = dm(r^2)$  which gives:

$$\int dI = \int (dm) r^2 = I = \int r^2 dm$$

Moment of inertia, is the rotational equivalent of a mass as the torque is rotational analogue of a force which we illustrated before. The total torque of a system is; the sum of the all torque contribution to the system. We know that:

$$F_{\text{tot},x} = ma_x$$

Then

$$\tau_{\text{tot},x} = I\alpha$$

And as

$$dk = F_{\text{tot},x}dx$$

Then

$$dk = \tau_{\text{tot},x}d\theta \quad \text{and} \quad \frac{dk}{dt} = \tau_{\text{tot}}\omega$$

The last one is happened since;

$$\frac{dk}{dt} = \frac{d}{dt} \left( \frac{1}{2} I \omega^2 \right) = I \frac{d\omega}{dt} \omega = \tau_{\text{tot}}\omega$$

By above review we write;

$$dk = \tau_1 d\theta + \tau_2 d\theta + \dots$$

To isolate one of the torque contributions, such as the first one we write:

$$\tau_1 d\theta = dk - \tau_2 d\theta - \tau_3 d\theta \dots$$

Thus the work expended for "changing the rotational kinetic energy", and for standing

against other torques. In a "steady rotation" we have  $dk = 0$ , then the work of one torque is done entirely against others.

If  $F_{\perp}$  be the perpendicular force exerted to a body and  $d\theta$  be the rotational equivalent of  $ds$  then since;

$$dw = rF_{\perp}d\theta$$

Which is implied that;

$$dw = \tau d\theta$$

We can write;

$$\tau = rF_{\perp}$$

The vector  $r$  is also called "the *lever arm* associated with the force  $F$ ". Thus torque also can be define as:

"A force equals the length of the lever arm, multiplied by perpendicular component of the force to the lever arm and to the axis of rotation".

If  $F_{tot} = 0$  and  $\tau_{tot} = 0$  then the rigid body is in translational and rotational equilibrium conditions respectively. This happen when;

$$F_{total} = ma, \tau_{total} = Ia, \text{ if } a = 0, \alpha = 0$$

*Geometrically; equilibrium under two forces requires the same line of action and under three coplanar forces they must be concurrent or parallel.*

Similar to linear motion, the center of mass (C) is also crucial in rotational motions. Under a zero

total force, point C undergoes uniform motion (spin). The total kinetic energy of an extended system is equal to the sum of translational and internal kinetic energies as;

$$K_{sys} = K_{tran} + K_{int}$$

Where  $K_{tran} = \frac{1}{2} m_{sys} v_c^2$  and  $K_{int}$  causes motion along the point C. in a rigid body, it is called rotational  $K_E$ .

In a rigid body the rotational motion depends on *angular momentum*. Thus we can summarize and introduce as follows:

<b>Angular momentum</b>	$L = I\omega$
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<b>Torque</b>	$\tau = rF_{\perp} = r_{\perp}F$
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<b>Kinetic energy</b>	$\mathcal{K} = \frac{1}{2} I\omega^2$
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<b>Work</b>	$d\mathcal{W} = \tau d\theta$
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<b>Power</b>	$\frac{d\mathcal{W}}{dt} = \tau\omega$
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Note is that; for a system of pivoting objects or bodies with a fixed common axis of rotation, which also have constant moments of inertia ( $I$ ) we have;

$$L_{sys} = \sum l_i$$

"The combined angular momentum of a system does not change unless there is a net external torque on the system". Thus when

$$L_{\text{syst}} = \text{const} \quad \text{if } \tau_{\text{tot}} = 0$$

is hold, then  $L$  will be conserved. This gives us the *law of conservation of angular momentum*.

When a revolving object changes its moment of inertia,  $I$ , then  $L = I\omega$ , does not apply. This is because there is no well-defined  $\omega$ . However even in this condition if the final and initial  $L$  be equal then one can define the conservative angular momentum because in this situation  $\tau = -\dot{L}$  and then  $\frac{dL_{\text{syst}}}{dt} = 0$ .

When central forces in a system depends on magnitude of  $r$  and not its direction we deal with *Isotropic central forces* i.e. when the function  $f(r) \propto |r| = r$ . Due to this the potential energy  $U$  is defined as:

$$U(r) = - \int f(r) dr$$

Thus the force is conservative:

$$\text{Iff } \exists U: f = -\nabla U$$

Where  $\nabla U = i \frac{\partial U}{\partial x} + j \frac{\partial U}{\partial y} + k \frac{\partial U}{\partial z}$  is the gradient. This works in conservative (frictionless in general) field of force.

Always  $U$  is existed thus *Isotropic central forces* are always conservative. They exert no torque thus angular momentum ( $L$ ) does not change i.e. it is constant. However in a gravitational and electrical field the energy which is a sum of potential energy and kinetic energy is also constant. We know that:

$$E = U + K \text{ and } L = m\omega r^2$$

Therefore one can build the equation of motion ( $F = ma$ ) as:

$$\begin{aligned} E &= U + k = U + 1/2(mv^2) \\ &= U + 1/2 m \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \omega^2 \right] \\ &= U + 1/2 m \left[ \left( \frac{dr}{dt} \right)^2 + L^2 / r^2 \omega^2 \right] \end{aligned}$$

We know that the general form of a linear differential equation of order  $n$  with constant coefficients is;

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots a_1 \frac{dx}{dt} + a_0 x = f(x)$$

Where each  $a_i$  is constant coefficients. In physics one can dare to say almost all nature can be describable with differential equations since all things is in motion. Sensible or not, deterministic or not, random or not, the whole *existence geometry* (nature) is in motion. All of this geometry is detectable and recognizable by the phenomena of *waves and vibrations*. The

simplest form to start is harmonic oscillations of a system of mass and string i.e. a 1-D pendulum thus, ignore friction and conservative. Since,  $E = U(x) + mv^2/2$  and is always constant, with  $\frac{dU}{dx} = -f$  then the time derivative of energy would be:

$$\frac{dE}{dt} = \frac{dU}{dx} \frac{dx}{dt} + mv \frac{dv}{dt} = -fv + mva = 0$$

However, the important point is that since the inverse relations are existed as:

$$\frac{dt}{dx} = \frac{1}{v} = -\sqrt{(E - U)2/m}$$

Then we have time as a function of position ( $x$ ) as:

$$t(x) = \int \frac{dx}{[E - U(x)2/m]^{1/2}}$$

With this review and the fact that  $\omega^2 = \frac{K}{m}$

Along with the calculation of last integration one can predict the position of a frictionless pendulum system as:

$$x = x_0 \cos(\omega t + c)$$

C is the constant. Thus the equation of motion ( $F = ma$ ) would be;  $ma + Kx = 0$ . The first order linear differential equation of motion is;  $m \frac{d^2x}{dx^2} + Kx = 0$  with the sin and cos as its solutions. According to these,  $U \propto \cos^2$  and  $K \propto$

$\sin^2$  hold their oscillations. And finally based on Euler identity ( $\cos^2 + \sin^2 = 1$ ) the amount of  $E = \text{constant}$ .

Generally friction (R) is proportional to velocity. Thus second order linear differential equation of motion for damped harmonic oscillator would be written as:

$$m \frac{d^2 x}{dt^2} + R \frac{dx}{dt} + Kx = 0$$

The solutions here would be:  $e^{i\theta} = \cos\theta + i \sin\theta$ . Since:

$$x = ae^{i\omega t}, \frac{dx}{dt} = i\omega t, \frac{d^2 x}{dt^2} = -\omega^2 x$$

Then we get the algebraic quadratic form as:

$$-m\omega^2 x + i\omega Rx + Kx = 0$$

With its famous solution transformed here as:

$$\omega = iR/2m \pm \left[ \frac{k}{m} - R^2/4m^2 \right]^{1/2}$$

When above harmonic oscillation is under the periodic friction/external force with same frequency ( $f$ ) of the oscillator we deal with *resonance effects*. The frequency obeys complex oscillation though  $f e^{i\omega t}$  is hold here. Therefore we have inhomogeneous differential equation of motion as:

$$m \frac{d^2 x}{dt^2} + R \frac{dx}{dt} + Kx = f e^{i\omega t}$$

With the solution of a quadratic algebraic equation transformed as:

$$-m\omega^2 a + Ri\omega a + Ka = f$$

Where  $a$  is the amplitude and  $x = ae^{i\omega t}$ . Note is that according to d'Alembert principle for equation of motion in the case of restrictive or constraining forces to a body or simply when the body is in contact with surface;  $\frac{dp}{dt} = \sum f$  holds for linear momentum (P) and  $\frac{dM}{dt} = \sum rxf$  holds for angular momentum (M). This is the "generalize the principle of virtual work" as long as forces of constraint occur. In fact with this principle we define our space of action. If the virtual displacement be  $\delta r$  then it say:

$$\sum_i \left( F_i - m_i \frac{d^2 r_i}{dt^2} \right) \delta r_i = 0$$

However in equilibrium when  $F\delta r = 0$ , this becomes:

$$\sum_i F_i \delta r_i = 0$$

Accordingly when  $F_i$  is conservative the total potential energy ( $U$ ) does not change i.e.  $\delta U = 0$ .

In a generalized point of view, all of the above discussions until now can be represented as something called *analytical mechanics* which one can say it is a bridge from Newtonian mechanics

and other developments of physics such as relativity and quantum mechanics.

Basically motion of a mechanical system is a function of time in space. This is the relative position of any point against a particular coordinate system i.e. frame of reference or reference frame. To define this function we need to something called degrees of freedom. It is the number of independent variable/parameters. For instance, the degrees of freedom for position of a system of  $N$  particles in Cartesian coordinate represented as  $3N$ . The point is that; in the case of distribution of points with fixed distances the number become less than the number of independent parameters  $3N$ . Since we are applying some constrains, our previous form of d'Alembert principle will become:

$$\sum_i \left( F_i - m_i \frac{d^2 x_i}{dt^2} \right) \delta x_i = 0$$

With  $x_i = x_1, x_2, x_3, \dots, x_{3N-1}, x_{3N}$  are coordinates of all  $N$  particles. Note is that  $x_i$  is a simpler representative of;

$$x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_N, y_N, z_N,$$

Therefore we need to generalize our coordinate which represented as  $q_i$  instead of  $x_i$  i.e.  $q_1 q_2 q_3 \dots q_i$ . These generalized coordinates

should satisfy the number of constraints for the system as well. For instance, in a system of two particle (a line) with fixed distance (as constrain) instead of  $6N$  degrees of freedom we have  $6N-1$  i.e. 5 degrees of freedom. For a system of three particles (a triangle) we have  $9N-3$  i.e. 6 degrees of freedom. The former is also true for a rigid body moving in space. Thus the independent parameters/variables define the position function of a mechanical system are called *generalized coordinates* represented as  $q_i, q_\mu$  or  $q_\alpha$  where  $i, \mu$  or  $\alpha$  are number of degree of freedom. According to newton second law; the equation of motion for a system of one particle will be:

$$m \frac{d^2 r}{dt^2} = F$$

Where  $F \equiv \sum \vec{F}_i$  and  $\frac{d^2 r}{dt^2} \equiv \vec{a}$  with the Cartesian components of  $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2}$  the mass  $m$  describe the features (characterize) of a particle. In fact Newtonian mechanics deal with masses and forces. Thus the second law of motion is about interactions among bodies in a mechanical point of view expressed in the form of differential equations with some limiting constraints. However since Newtonian mechanics says "giving the present, predict the future", the *past*

configuration or arrangements among bodies (particles) does not be considered. The mass is constant in classical mechanics. The acceleration of a set of particles/bodies is proportional to the interactions among them relative to an inertial frame of reference i.e. a reference frame in which Newton third law is satisfied. For instance Earth is not an inertial reference frame since it keeps rotating. For a free (not interacted) particle moves with respect to inertial frame of reference linearly and uniformly thus by "its own momentum". But in nature there are *action of forces* on bodies/particles. It means they are in direct contact with each other or with surfaces cause *forces of friction*. The crucial role of such forces is that they transit macroscopic motions of a body into microscopic motions of its molecules and atoms represented as *heat* generation. Thus we reduce this forces of interactions to (average) frictions and heat. However for rigid bodies this forces of interaction are reduced to "kinematic properties of rigid constraints" which "force the particles to move on definite surfaces". Therefore there is no need to consider microscopic interactions and one can define the equation of motion for such bodies by their "own

macroscopic generalized coordinates"  $q_\alpha (1 \leq \alpha \leq v)$ .

The acceleration is due to this *reaction forces of rigid constraints* is a curvilinear motion since velocity is a vector quantity. However since these constraints does not perform any work on the system, they do change on the direction and not on the magnitude of the velocity. This is because doing work on a system generate heat and then change it kinetic energy but this does not imply by this *ideal rigid constraints* due to the perpendicularity of these forces with the displacement of the system. And thus to the direction of particle instantaneous velocity. However in calculation usually ignore these forces from start entirely since the number of generalized coordinates, is equal to the number of degrees of freedom of a system.

Generally Cartesian coordinates denoted by  $x_i$  with  $i$  varies from 1 to  $3N$  i.e. from 1 to  $n$ . Thus the generalized coordinate, determine the position of the system. In fact we deal with functions of many variables/parameters as:

$$x_i = x_i(q_1, q_2, q_3 \dots q_\alpha \dots q_v).$$

Differentiating this with respect to time give:

$$\frac{dx_i}{dt} = \sum_{\alpha=1}^v \frac{\partial x_i}{\partial q_\alpha} \frac{dq_\alpha}{dt}$$

Or simply as

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_\alpha} \frac{dq_\alpha}{dt}$$

Another notations are:

$$\frac{dx_i}{dt} \equiv \dot{x}_i, \quad \frac{dq_\alpha}{dt} \equiv \dot{q}_\alpha$$

Thus we can write:  $\dot{x}_i = \frac{\partial x_i}{\partial q_\alpha} \dot{q}_\alpha$  for the velocity and differentiation this again gives acceleration as:

$$\ddot{x}_i = \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_\alpha} \right) \dot{q}_\alpha + \left( \frac{\partial x_i}{\partial q_\alpha} \right) \ddot{q}_\alpha$$

However if one consider  $q_\alpha$  as  $q_\beta$  for the acceleration this would become:

$$\ddot{x}_i = \frac{\partial^2 x_i}{\partial q_\alpha \partial q_\beta} \dot{q}_\alpha \dot{q}_\beta + \left( \frac{\partial x_i}{\partial q_\alpha} \right) \ddot{q}_\alpha$$

According to our defined Cartesian and generalized coordinates above we can now consider three components of a force vector applying on a particle as a scalar function U or V. Thus the Newton's second law transform as:

$$F_i = - \frac{\partial U}{\partial x_i}$$

For the *attraction*, *electrostatic* and *elastic* forces which later come along. U is called the *potential energy* of a system as well. Therefore one can write the equation of motion based on

**d'Alembert principle** for the reaction forces ( $\sum_{i=1}^n F'_i dx_i = 0$ ) and potential forces  $U$  as:

$$m_i \left( \frac{\partial^2 x_i}{\partial q_\alpha} q_\alpha \frac{\partial^2 x_i}{\partial q_\beta} q_\beta + \left( \frac{\partial x_i}{\partial q_\alpha} \right) \ddot{q}_\alpha \right) = - \frac{\partial U}{\partial x_i} + F'_i$$

If one express the generalized coordinate for the potential energy it would be as:

$$U = U(q_1, q_2 \dots q_\alpha \dots q_v)$$

And if one substitute the velocities ( $\dot{x}_i$ ) in the expression of kinetic energy ( $T$ ) he/she get:

$$T = \frac{1}{2} \sum_{i=1}^n m_i \dot{x}_i^2$$

Thus the kinetic energy is a function of generalized coordinate ( $q_\alpha$ ) and generalized velocities ( $\dot{q}_\alpha$ ). The difference between kinetic and potential energy is called the **Lagrangian function**;

$$L = \sum_i \frac{1}{2} m_i v_i^2 - U(x_1, x_2 \dots x_{3N})$$

Or simply as;  $L = T - U$ . And finally we get *Lagrangian equation* as:  $\frac{d}{dt} \frac{dL}{dq_\alpha} - \frac{\partial L}{\partial q_\alpha}$  with  $1 \leq q \leq v$  and according to the number of degrees of freedom. This is a function of  $q_\alpha$  and its time derivatives as;  $\frac{d[\partial L / \partial \dot{q}_\alpha]}{dt} = \frac{\partial L}{\partial q_\alpha}$  which is imply:

$$m_i \frac{dv_i}{dt} = - \frac{\partial U}{\partial x_i}$$

In practice above formulas will change according to the problem. For instance for a pendulum of length  $l$  the general coordinate is the angle  $\theta$ ,  $T = 1/2mv^2 = 1/2ml^2(\frac{d\theta}{dt})^2$  and also  $U = -mgl \cos\theta$ . The Lagrangian of this system is:

$$L = 1/2ml^2\dot{\theta}^2 + mgl \cos\theta$$

According to d'Alembert principle in the lagrangian form we will have:

$$ml^2 \frac{d^2\theta}{dt^2} = -mgl \sin\theta$$

In spherical coordinate the lagrangian functions is expressed as:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2 - U(r))$$

And the lagrangian equation would be:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}$$

Which yield us:

$$\frac{\partial L}{\partial r} = m\dot{\theta}^2 + m\dot{\phi}^2 \sin^2\theta - \frac{\partial U}{\partial r},$$

$$\frac{\partial L}{\partial \theta} = mr^2 \sin\theta\cos\theta\dot{\phi}^2 \text{ and } \frac{\partial L}{\partial \phi} = 0$$

According to the lagrangian the conservation laws of a system will also change with the help of

integration on the equation of motion as in the case of energy:

$$E = \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L, \frac{dE}{dt} = - \frac{\partial L}{\partial t}$$

When  $\frac{\partial L}{\partial t} = 0$  hold i.e. when L does not explicitly depends on time. In other words, d'Alembert principle is satisfied with having constant constraints and constant external forces. Thus this is an integral of motion is applied, the energy is conserved and the system is closed though.

If this is not hold then  $E = T + U$  and since U does not depend on velocity but on r then we write:  $\frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha}$  which implies that:

$$E = \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L$$

The generalized momentum in terms of generalized coordinate expressed as:

$$P_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha} = \text{const}; \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha}$$

With  $P_x = mv_x = \frac{\partial L}{\partial v_x}$ . For a system of N particles we calculate center of mass (R) instead of every individual particle though.

$$P = \frac{\partial L}{\partial \dot{R}} = \left( \sum_{i=1}^N m_i \right) \dot{R} = \text{const}$$

Thus the total momentum is also conserved.

For a plane rotational motion in a generalized polar coordinate  $(\theta, r; r = (x^2 + y^2 + z^2)^{1/2})$  the momentum i.e. the angular momentum ( $M = \vec{r} \cdot \vec{p}$ ) as a vector product of radius vector and momentum. Geometrically we have parallelogram with area of  $A = rp \sin \alpha$  and  $\alpha = \text{angle between } r \text{ and } p$ . The angular momentum for a system of particles is  $M = \sum_{i=1}^N \vec{r}_i \vec{p}_i$ . When there is no external forces i.e.  $\dot{\vec{p}} = 0$  and  $\dot{\vec{r}}$  is in the same direction of  $\vec{p}$  and their vector product is equal to zero so then  $\frac{dM}{dt} = 0$  and  $M$  is conserved. This is also true for relative motion when  $\frac{dM'}{dt} = 0$ . In the case of angular momentum as a generalized momentum ( $P_\theta$ ) in polar coordinate where theta is the angle of rotation around polar axis (z) would be as:

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = M_z$$

With  $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r); \theta = 0, \theta = 1$

Thus a closed system i.e. a system without external forces satisfies d'Alembert principle to have integral of motions in three vector components of linear and angular momentum along with the energy.

In a more clarified and simple way (in 1-Dimension) one can say; the law of motion ( $x = x(t)$ ) must satisfy  $m\ddot{x} = f(x, t)$ ; the differential equation. If one denote  $U(x, t)$  and calculate its antiderivative with respect to  $x$  with  $-f(x, t)$  as:

$$U(x, t) = - \int_{x_0}^x f(x, t) dx$$

$U(x, t)$ , is the potential of the given field of force.

We also have  $\frac{m\dot{x}^2}{2}$ , as the kinetic energy of the point of mass. Since one can write the differential equation as:

$$-\frac{dU}{dx} - \frac{d}{dt} \frac{dT}{dx} = 0.$$

And then we can write;

$$L(t, x, \dot{x}) = T - U$$

Thus we can name lagrangian function as *kinetic potential* function. It is clear that,  $U$  does not depend on  $\frac{dx}{dt} \equiv \dot{x}$ . This is also true for  $T$  with respect to  $x$ . thus instead of  $(-\frac{dU}{dx} - \frac{d}{dt} \frac{dT}{dx} = 0)$  one can write;

$$\frac{dL}{dt} - \frac{d}{dt} \frac{dL}{d\dot{x}} = 0$$

This is the **Euler's equation** corresponding to functional:

$$w = \int_{t_0}^{t_1} L(t, x, \dot{x}) dt,$$

Which is called *Hamiltonian principle of least action*. Where  $x(t_0) = x_0$  and  $x(t_1) = x_1$  represent initial and terminal fixed end points of the motion path. In generalized coordinate the action ( $W$ ) is either a maximum or a minimum which can also be written as:

$$w = \int_{t_0}^{t_1} L(t, q_\alpha, \dot{q}_\alpha) dt$$

In fact, Hamilton's principle say; in sufficiently very small intervals of time the value of integral is a minimum stationary value while for long time intervals the action value is a minimax. Thus we seek for stationary value in general. This is a familiar way in *calculus of variations* which more deals with extrema of functional (a class of functions attributed to a certain numerical values) in infinite number of degrees of freedom. However the variation ( $\delta$ ) is the same as in the d'Alembert principle.

While Lagrange function ( $L$ ) proposes total energy as the difference between kinetic and potential energies, the Hamiltonian function represent this as the sum of these two.  $H = T + U$ , which is the function of generalized coordinates ( $q$ ) a momenta ( $p$ ). In other words, for

a particle in a simple 1-D space  $L(x, v)$  while  $H(x, v)$ . The generalize momentum would represent as:  $P_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$  applying constrains and  $p = m \frac{dq_\alpha}{dt}$  without them. Thus  $\frac{\partial L}{\partial q_\alpha} = \frac{dp_\alpha}{dt}$  is another form of Lagrange equation. Hamiltonian function express energy with respect to coordinates and momenta. This is the same as Lagrange function but the velocity is replaced by the momenta i.e.  $E(q, \dot{q},) \equiv H(q, p)$ . However one can say:  $H(q, p) = q_\alpha p - L$ .

Remember, we wrote the time derivative of Lagrange as:

$$\begin{aligned} \frac{dL}{dt} &= \sum_\alpha \frac{\partial L}{\partial q_\alpha} \frac{dq_\alpha}{dt} + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d\dot{q}_\alpha}{dt} = \sum_\alpha \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} \frac{dq_\alpha}{dt} + \\ \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d\dot{q}_\alpha}{dt} &= \frac{d}{dt} \left[ \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \right] \end{aligned}$$

Thus one can with the energy as:

$$E = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$$

Therefore,  $\frac{dE}{dt} = 0$  and energy is constant. More simply we can write;

$$\begin{aligned} H &= -L + \sum \dot{q}_\alpha p_i = -(T - U) + \sum \dot{q}_\alpha \frac{dT}{\dot{q}_\alpha} = \\ &-(T - U) + 2T = T + U. \end{aligned}$$

Thus again we say that; H is the total energy of the system. In an autonomous system i.e. a system

whose potential energy and constraints do not depend on time  $t$ . In fact they are neglected. Thus according to Hamiltonian principle,  $H$  is constant. Consequently:

"The total energy of an autonomous canonical (Hamiltonian) system remains constant in the process of motion". This system is called *conservative system*. The system which does not hold these properties is called *dissipative system*. Analogous with  $dL$  there is:

$$dH = \sum_{\alpha} \left( \frac{\partial H}{\partial q_{\alpha}} \right) dq_{\alpha} + \left( \frac{\partial H}{\partial p_{\alpha}} \right) dp_{\alpha}$$

By this and our previous discussion one can find the canonical equations as:

$$\dot{p}_{\alpha} = -\frac{\partial H}{\partial q_{\alpha}}, \dot{q}_{\alpha} = \frac{\partial H}{\partial p_{\alpha}}, \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

The generalized coordinate and momenta ( $q_{\alpha}$  and  $p_{\alpha}$ ) are *canonical variables*. Literally canonical means; simple and symmetric. However this is also much related since if we neglect  $q_{\alpha}$  in  $L$  then it is invariant with respect to changes in  $q_{\alpha}$ ; the continuous variable which leads to a constant momentum  $p_{\alpha}$ . Emmy Noether in 1918; using canonical forms, proposed and proved that: "For every invariance with respect to a continuous variable  $q_{\alpha}$  there accordingly is a conservation law". Thus for the whole system; the

invariance of the system's total rotation the invariance of angular momentum and the invariance of the system's total translation the invariance of the total momentum.

For instance a free particle in 1-D space subject to an elastic force  $kx$ ; i.e. a harmonic oscillator has the Hamiltonian function as;

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2 = T + V$$

Given two physical quantities as two functions  $F(q, p, t), G(q, p, t)$  then the Hamiltonian would be:

$$\{F, G\} = \sum_i \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

This is called the *Poisson brackets*. If " $F$  is not explicitly time dependent, its being a constant of motion is equivalent to the vanishing of its Poisson bracket with the Hamiltonian". However by the chain rule of differentiation the time derivative of  $F$  would be:

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t},$$

"Every object in universe attracts every object", this is the *law of gravitation* which is uniform in near distances such as Earth's gravity acceleration

(*g*) and non-uniform in large distances such as astronomical scales. In astronomical scales we have inverse-square distance law. Because of the presence in very tiny (quantum) scales to very large (General Relativity) scales, one can say gravity is the most fundamental law of physics.

*Distances* and *periods* of orbiting are two crucial basics for creation of the theory of universal gravitation. In  $F=G Mm/r^2$  we have a combination of them to give us the first important generalization in physics, i.e. *universal gravitation*.

*Triangulation* which is a method to construct a triangle by a fix point and two directions along with the laser and radar ranging helps scientists to detect *distances* in space. For periods we should have "accurate timing", exact observations and a "proper interpretation" of them. Bellow we have some astronomical data to clarify our mean.

	Mercury	Venus	Earth	Mars
Distance from Sun (m)	$5.79 \times 10^{10}$	$1.08 \times 10^{11}$	$1.50 \times 10^{11}$	$2.28 \times 10^{11}$
Radius (m)	$2.44 \times 10^6$	$6.05 \times 10^6$	$6.37 \times 10^6$	$3.39 \times 10^6$
Mass (kg)	$3.30 \times 10^{23}$	$4.87 \times 10^{24}$	$5.97 \times 10^{24}$	$6.42 \times 10^{23}$
Density ( $\frac{\text{kg}}{\text{m}^3}$ )	5430	5250	5520	3950
Orbit (years)	0.24	0.62	1.00	1.88
Rotation Period (hours)	1408	5832	23.9	24.6
Tilt of axis	2°	177.3°	23.5°	25.2°

Radius of the sun (m)	$6.96 \times 10^8$
Mass of the sun (kg)	$1.99 \times 10^{30}$
Radius of the moon (m)	$1.74 \times 10^6$
Mass of the moon (kg)	$7.35 \times 10^{22}$
Distance of moon from Earth (m)	$3.84 \times 10^8$

Gravity is a force of attraction between two objects because of their mass. The cause of this pull is not known. The more mass the stronger force, but the force gets weaker as the object gets farther away. Thus, the gravitational attraction of a single object is *directly proportional to its mass, and inversely proportional to its distance*.

$$F_g \propto \frac{m}{d}$$

For two objects we have;

$$F_{g,1} \propto \frac{m_1}{d_1} \quad \text{and} \quad F_{g,2} \propto \frac{m_2}{d_2}$$

Because each object attracts other one, the total force would be:

$$F_g \propto \frac{m_1}{d_1} \cdot \frac{m_2}{d_2} = \frac{m_1 m_2}{d^2}$$

However for extended objects (a system of particles) which pull another particle, P, we need a *superposition principle*. This means that "The combined gravitational force on P is the vector sum of the separate forces due to A, B, ... , each taken in isolation with P."

However, the exerted gravity on any object A by a spherically symmetric object B is equal to the whole center of mass of B. This also holds for two spherically symmetric objects A and B. But why the inverse square law is applied here? Because;

$$F \propto \frac{1}{r^2}$$

Another point is that, the law is universal since it calculates the total exerted force of masses on each other, so no matter they are point masses  $m$  or planets, moon etc. In fact this follows from the law of *action and reaction*. Therefore the moon does fall toward the earth but not on earth.

What is  $G$ ? The gravitational constant! "It governs the behavior of all matter". Like mass, it cannot directly be measured. Therefore we write:

$$G = \frac{r^2 F}{mM}$$

Therefore we need delicate experiments, due to extreme weakness of the gravitational attraction between non-celestial objects. Henry Cavendish (1798), has done this with two spheres of known  $M$ ,  $m$ ,  $r$  and measured  $F$ . This can also be applied to planetary motion.

Suppose we have circular motion for a planet of mass  $m$ , then from Newton's second law;  $F = ma$

where  $a = \left(\frac{2\pi}{T}\right)^2 r$  and  $F_{tot} = \frac{GM_s m}{r^2}$ , where  $M_s$  is mass of the sun, we have:

$$\frac{GM_s m}{r^2} = m \left(\frac{2\pi}{T}\right)^2 r$$

Because we seek for, relating  $T$  and  $r$ ; then:

$$\frac{r^3}{T^2} = \frac{GM_s}{(2\pi)^2}$$

Where

$$M_s = \left(\frac{2\pi}{T}\right)^2 \frac{r^3}{G}$$

Since the right hand side have no useage here , we are left with  $r^3/r^2$ , which is the Kepler's third law: "The square of a planet's period of revolution is proportional to the cube of its distance from the Sun" i.e  $T^2 \propto r^3$

therefore there is no circular orbits in planetary motions but there is elliptic orbits (Kepler's Rules) dominated their motions. Remember that *All three Kepler rules are consequences of Newton's three laws.*

Gravitational forces differ with position ( $dr$ ). Along with the gravitational force, there must be a potential energy called Gravitationa potential energy ( $E_p$ ) thus:

$$F_G \cdot dr = -dE_p$$

Consequently, the force of gravity is conservative if above position function is hold. Where, for two particle of mass  $M$  and  $m$  we have;

$$E_p = -\frac{GMm}{r}$$

This is an inverse-distance ( $1/r$ ) function i.e.  $E_p$  increase according to the distance from the center of gravity (central mass). Thus we need positive work against gravity to lift an object. Our reference level is when  $E_p = 0$ , in fact at infinite distance from center of attraction i.e. when  $r \rightarrow \infty$ . For earth;  $E_p = mgz$  where  $z$  is the altitude.

Mathematically one can prove the inverse-distance formula by combining and working on two above equations as:

$$dE_p = d\left(-\frac{GMm}{r}\right) = \frac{GMm}{r^2} dr$$

And also;

$$-F_G \cdot dr = -\frac{GMm}{r^2} dr$$

However one can get this inverse-distance ( $1/r$ ) potential energy from  $F = \frac{GMm}{r^2}$  by integration.

Since  $F$  is conservative, for path of a particle under gravity from point A down to point B we have;

$$\begin{aligned}
E_{p_A} - E_{p_B} &= \int_A^B F \cdot dr \\
&= \int_A^X F \cdot dr + \int_X^B F \cdot dr \\
&= - \int_{r_A}^{r_B} \frac{GMm}{r^2} dr
\end{aligned}$$

The first integral is removed since  $F$  is transverse;  $F \cdot dr = 0$ . In the second one  $F$  and  $dr$  goes to  $M$  and  $-dr$  is positive as well. Therefore since;

$$F \cdot dr = F |dr| = (F)(-dr)$$

We continue as:

$$= \frac{GMm}{r_B} - \frac{GMm}{r_A}$$

because,  $A$  was a fixed reference point, where  $E_p = 0$  and  $B$  was an arbitrary point where  $E_B = E_p$  as  $r_B = r$  we write:

$$\begin{aligned}
E_p &= -\frac{GMm}{r} + \frac{GMm}{r_A} \\
E_p &= -\frac{GMm}{r} \quad \text{as } r_A \rightarrow \infty
\end{aligned}$$

For pairs ( $ij$ ) of particles -as in previous cases- we have a sum of mutual gravitational potential energies ( $E_{p_i}$ ) as:

$$\sum_{ij} -\frac{Gm_i m_j}{r_{ij}}$$

However, in the case of intense gravitation beyond the solar system such as in black holes and in the velocities up to the speed of light, Newton theory of gravitation does not comply with such cases. Thus we need a stronger theory by Einstein to overcome these problems. The theory of *spacial and general relativity*.

We have learned that, Newton's second law can be written as;  $F = d(mv)/dt$  where  $m$  is assumed constant. Today this is not true since we now know that; *the increase of the velocity the increase of the mass of object*. Einstein corrected this as:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

Where  $m_0$  is the "rest mass" or mass without motion. In fact that's all important changes to the Newton second law which introduced Special Theory of Relativity in 1905. Extending the STR to the Newton law of gravitation; Einstein introduced General Theory of relativity (GTR).

However the principle of relativity by itself is happen to you when you are traveling in a vehicle with a uniform velocity and do not look outside, then you do not have a sense of moving for the vehicle! Newton explained it as "The motions of bodies included in a given space are the same

among themselves, whether that space is at rest or moves uniformly forward in a straight line."

We know from Galilean relativity that; if a particle move with a uniform velocity ( $v$ ) from a point  $x$  to  $x'$  in  $x$ -direction we write:

$$x' = x - vt, y' = y, z' = z, t' = t$$

putting this transformation of coordinates into Newton's laws we find that they leave in the same form for a moving system and a stationary system. Thus one cannot have sense whether the system is moving or not.

Maxwell's equations of the electromagnetic field, describe electricity, magnetism, and light in a system of four partial differential equations. If one substitute above Galilean transformations into them lead to a conclusion that, *their forms does change*. Thus the transformation does not comply with the Maxwell's equations to hold Newton second law in the case EM waves and field. EM waves move at the same speed ( $c$ ) independent of the source of generation being moving or stationary. The former is special for EMW the latter holds for the sound waves as well.

To solve this problem there was a need for another transformation instead of Galilean one to comply with Maxwell's laws of electrodynamics. H. A.

Lorentz proposed a transformation called Lorentz transformation as:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, y' = y, z' = z, t' = \frac{t - vt/c^2}{\sqrt{1 - v^2/c^2}}$$

They are transforming equations in terms of behaviour of bodies in one frame of reference to another. They are hold, If  $x$  is taken in the direction of the relative velocity of the two frames, and  $v$  is the magnitude of the relative velocity. thus time is not a universal constant and that space-time is a 4-D structure in the case of light or close to the velocity of light.

Applying this transformations to Maxwell's equations of electrodynamics resulted no change to the forms of the equations. Then Einstein, proposed that "all the physical laws should be of such a kind that they remain unchanged under a Lorentz transformation". Therefore Einstein rewrite Newton's equations of mechanics, Maxwell's laws of Electrodynamics left unchanged and the old idea of space and time replaced with new peculiar idea of space-time. The first is happed by substituting the constant  $m$  with;

$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$  ,then these two laws will synchronize.

Through Michelson-Morley experiment - which rejected the existence of hypothetical medium called *ether* by showing that on earth the velocity of light is constant everywhere and proved that the value of  $c$  is the same in all frames of reference in vacuum- Lorentz suggested that, "material bodies contract when they are moving," only in the direction of motion. But Einstein showed that this is true when the time is also slowed down i.e. a contraction of time.

Remember that;

the corrections brought about by relativity are trivial for objects whose speeds are small compared to that of light i.e. when  $v \ll c$ . However, for particles moving at velocity near to speed of light such as in atomic, nuclear, and astrophysical sciences; relativity is an inseparable part. Altogether the principle of relativity says:

**"The laws of physics are the same in all inertial systems".**

However it is not to say that all matters are relative or there is no absolute velocity in universe though! According to Big Bang theory, "more than  $10^{10}$  years ago, electromagnetic waves still survive", and even exert a so-called 3-Kelvin background *radiation pressure*. This radiation

varies in strength and propagates from different directions of the cosmos. For instance our mother earth "moves at several hundred kilometres per second relative to this radiation".

To continue we need a new concept called *lorentz force* which is the "total electromagnetic force on a charge". Its formula is:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Where  $q$  is the charge,  $E$  is the *electric field* due to existence of electric force independent of the motion of the charge and  $B$  is the *magnetic field* due to magnetic force, depends on the direction and velocity of the charge. Thus "every point in space is characterized by two vector quantities which determine the force on any charge". For our purpose we write this formula as:

$$\mathbf{F}/q = \mathbf{E} + \mathbf{v} \times \mathbf{B}/c$$

Now we have a "general law of nature and therefore valid in all inertial systems". But wait; the velocity from where must be calculated? The answer is in the theory of electrodynamics of moving bodies known as STR. Thus all is happening in relativistic electromagnetic fields. The classical transformation of position and time was describe before as:

$$x = x - vt, \quad t = t.$$

According to Galilean transformation, *Addition of velocities* says; the velocity of light waves emitted from a body moving with velocity  $v$  must be:  $c \pm v$ . But according to Lorentz transformation and also trials such as Michelson-Morley experiments, it is proved that beside additive property it is also have the following properties.

1. The light has constant velocity  $c$  in vacuum. i.e.

$$x' = ct' \text{ if } x = ct$$

2. In all inertial systems we have uniform motions in a straight line if the motion is force-free and un-accelerated. i.e. linear transformation of position and time.

$$x = ax - bt \quad t = Ax - Bt$$

Where  $a$ ,  $b$ ,  $A$  and  $B$  are transformation coefficients.

3. There is no preferred reference system for position and time transformations. Thus if one know the values of transformation coefficients relative to velocity  $+v$ , then is able to find reverse transformations relative to velocity  $-v$  i.e.

$$x = x(x', t') \quad t = t(x', t')$$

If  $v = \sqrt{v_x^2 + v_y^2 + v_z^2} = c$  then  $v' = c$  as well. Thus light velocity does not alter passing through inertial system.

Suppose we have system moves with velocity  $v$  relative to the other. In the origin we have:

$$x = vt$$

Consequently,

$$x = 0 = avt - bt, \quad \text{then } b = av.$$

Therefore

$$x = a(x - vt)$$

And according to third property

$$x = a(x + vt).$$

Thus instead of first property based on additive property we write:

$$\text{If } ct = at(c + v), \Rightarrow ct = at(c - v),$$

$$ct' = a \left[ \frac{at'(c + v)}{c} \right] (c - v)$$

Or simply

$$a = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

This result is called *time dilation* which is commonly denoted by  $\gamma$ . Therefore:

$$\begin{aligned} b &= v\gamma & x &= \gamma(x - vt) \\ x &= \gamma(x' + vt') = \gamma(\gamma x - \gamma vt + vt') \\ t' &= \gamma t + \frac{x(1 - \gamma^2)}{\gamma v} = \gamma(t - vx/c^2) \end{aligned}$$

These also set coefficients A and B and the final result is the Lorentz transformation as;

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - vx/c^2)$$

In 3-D we add  $y = y'$  and  $z = z'$ .

Because  $c$  is consistent thus:  $x^2 - c^2t^2 = x'^2 - c^2t'^2$  is hold. In 3D we write  $r^2 - c^2t^2 = r'^2 - c^2t'^2$ . Thus does not change under Lorentz transformation and we have an invariant quaintly called *Lorentz Invariant* as bellow:

$r^2 - c^2t^2 = 0$  : When propagation is equal to the velocity of light. This is called *light-like* cause and effect relation.

$r^2 - c^2t^2 > 0$ : When propagation is greater than the velocity of light. This is called *space-like* cause and effect relation.

$r^2 - c^2t^2 < 0$  : When propagation is more slowly than the velocity of light. This is called *time-like* cause and effect relation.

Why according to STR the travelers stay young in journey to outer space than people on earth? Because of contraction of time in the speed near to the speed of light. More precisely; if on the origin point one put an accurate clock in the  $x'$  system, which moves with velocity  $v$  relative to the  $x$  system of the observer then we write:

$$t' = \gamma \left( t - \frac{v^2 t}{c^2} \right) = t \sqrt{(1 - v^2/c^2)}, \text{ if } t' < t$$

Since at origin  $x' = 0$  and  $x = vt$  or generally  $t = \gamma(t' + vx/c^2)$  since  $x' = 0$ , then  $t = \gamma t'$ .

Remember that, time dilatation and length contraction can be linked as:  $\gamma = \frac{1}{[1-v^2/c^2]^{1/2}}$ .

Thus despite the separate alteration of length and time, by the forth equation of Lorentz transformations we can construct a space-time interval between two events occur on the abscissa (x-axis, length;  $l$ ) and ordinate (y-axis, light-time;  $ct$ ) coordinates. More precisely, we see events simultaneously occur at two separated places,  $x$  and  $x'$ . But this is not true. Since this is observed by an observer at position  $x$  not  $x'$ . Thus they differ in time. If we have position 1 and its corresponding position 2 then, this can be showed as:

$$t'_2 - t'_1 = \frac{v(x_1 - x_2)/c^2}{[1 - v^2/c^2]^{1/2}}$$

Finally we note that:

The time interval between to events is called *proper time* and represented as:

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - (v/c)^2}} = \gamma \Delta \tau$$

From above it is cleared that, the Lorentz transformation is a core in STR and is also analogous to rotation of  $r^2$  around its axis, but in space and time. One can shows this in 3-D as:

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2.$$

However if one consider this transformation as transforming of vector components and coordinates which stay unchanged then we will have vectors with three space-components and one time component, i.e. a 4-D space. This pave the way to relativistic electrodynamics and its famous formula of  $E = mc^2$ .

From Newton's second law  $F = ma$  and its time alterations we get  $F = \frac{dp}{dt} = \frac{dmv}{dt}$  i.e. force is the rate of change of momentum. Using  $m_0$  instead, one get;

$$\mathbf{p} = m\mathbf{v} = \frac{m_0 \mathbf{v}}{[1 - v^2/c^2]^{1/2}}$$

Thus if action and reaction are still equal the conservation of momentum is still hold in the same way as before despite the fact that mass is not constant any more.

But we know that in classical mechanics, momentum is proportional to velocity. It can be hold in this *relativistic mechanics* as well. Because in considerably  $v < c$  and sufficiently low inertia the value of  $\sqrt{1 - v^2/c^2}$  asymptotically approaches to 1 then  $p \propto v$ . While when,  $v \cong c$  and sufficiently high inertia the value of  $\sqrt{1 - v^2/c^2}$  asymptotically approaches

zero then  $p \rightarrow \infty$ . Therefore *in relativistic mechanics we have continuous increase of momentum since the mass is increasing.*

In the case of long time and constant exerting force on a body classical mechanics says; the velocity continually increase and goes to infinity since the momentum depends on velocity. While in relativistic mechanics this happens for momentum, since the mass is increasing. Even when  $v = c$  the momentum continues to increase. Finally instead of;  $m = m_0/[1 - v^2/c^2]^{1/2}$

Because of fall of classical constant mass in relativistic mechanics, Einstein proposed that;  $m = \frac{E_{total}}{c^2}$ . In the case of Kinetic energy one can write;  $m = \frac{E_{kinetic}}{c^2}$ . Since the mass is not constant

one can also write;  $\Delta m = \frac{\Delta E}{c^2}$  or more clearly as:  $m \cong m_0 + \frac{1}{2} m_0 v^2 / c^2$ . This is originated from the fact that in the case of gas and molecules increase in temperature leads to the increase of velocity of molecules and this cause increase of mass.

One can expand the last result as:

$$mc^2 = m_0c^2 + \frac{1}{2}m_0v^2 + \dots$$

Since  $m_0c^2$  is the "rest energy" or energy in inertial point and  $\frac{1}{2}m_0v^2$  is the K.E thus Einstein proposed that:

$$E = mc^2 \text{ or } \Delta E = \Delta mc^2$$

This famous formula represent the equivalence of mass and energy in relativistic point of view to the universe. One can prove this by exerting a force to body from  $E = m_0c^2$  i.e. when the body is at rest. Consequently the body start to move and its K.E increase which leads to increment of its mass. Thus the continuous force (F) the continuous increase of  $m$  and  $E$ .

From the fact that;

$$\frac{dT}{dt} = \frac{d}{dt} \left( \frac{1}{2}mv^2 \right) = \frac{1}{2}m2v \frac{dv}{dt} = mv \frac{dv}{dt},$$

Where T is another symbol for Energy, together with Newton's second law;  $F = ma$ ;  $a = \frac{dv}{dt}$  one can write:  $\frac{dE}{dt} = F \cdot v$  which in its relativistic form will be represented as:

$$\frac{dmc^2}{dt} = \frac{dmv}{dt} \cdot v$$

With some algebraic tricks and simple differentiating on this relation, one can obtain the relativistic mass as before:

$$m = m_0/(1 - v^2/c^2)^{1/2}$$

This relation has a central role in equivalence an agreement of mass and energy in STR. However we saw that there also almost such a relation between mass and momentum. To put all together we need some retardations.

"A point that moves inertially, that is, a point that moves uniformly in a straight line in space is represented by a straight line in a space-time. For such a motion covers the same distance in the same time."

Being invariant, the velocity of light can be represented geometrically in space-time geometry using a structure called "light cone" since  $l = ct$ . In a Lorentz space greater than 2-D the set of all space-like radius vectors is separated from that of the time-like vectors by the light cone formed of all vectors of zero pseudo length i.e.  $\sqrt{x \cdot x}$ . If  $x$  is a vector the former set of points is connected, i.e. all the space-like directions are equivalent. The latter set consists of two separate connected components i.e. time is *irreversible*. Thus in any frame of reference traveling to the past is impossible.

However since in relativistic mechanics we deal with *set* of transforming-but invariant- vectors then mathematically we have a combination of linear algebra, set theory and calculus. This is called *tensors* in mathematics. In fact it is the true

sublanguage of this topic in both STR and GTR along with the associated geometry.

In a relativistic point of view, the fields E and B are only different displays of the same original field. They are the 6 components of a 4-D *antisymmetric field tensor*.

We have used the 4-D scalar product  $r^2 - c^2t^2$ . Thus we can now also define an imaginary length  $l_4 = ict$ . Note is that  $i$  is used to show oscillations and also to produce required minus sign. This is the 4<sup>th</sup> component in space time vector  $(x, y, z, ict)$ . Thus we are in a pseudo-Euclidian spaces i.e. a space which does not hold the condition of  $x \cdot x > 0$  for any  $x \neq 0$  in a real linear space. Therefore instead of ordinary scalar product we have pseudo-scalar product defined in a 2-D *Lorentz plane* as;

$$xy = \sum_i x_i y_i \quad i = 1, 2, 3, 4$$

This is called Minkowski 4-D space-time field after him. However, Euclidian distance between two points in a plane is also independent of the coordinates used to describe those points. Thus it is an invariant geometric quantity as well.

According to GTR big masses curve the space. This can be represented by a metric tensor  $g_{ij}$  in Minkowski space which is  $(1, 1, 1, -1)$  along the

main diagonal and zero elsewhere as long as we deal with STR. Then the space-time vector is (x, y, z, ct). Thus our scalar product alter to;

$$xy = \sum_{ij} x_i g_{ij} y_j$$

Thus we can write  $l_i = (r, ict)$  in general and transforming in a Lorentz plane as position-time vectors. However scalar relation such as  $\sum_i l_i^2 = r^2 - c^2 t^2$  would not change. Differentiating length in terms of time as;  $\frac{dl_i}{dt}$  do not yield a 4-D vector since  $t$  is not a scalar. However,  $r^2 = t^2 - r^2/c^2$  is a scalar, and its differential would be;

$$dr = [dt^2 - dr^2/c^2]^{1/2} = \left[ - \sum dl_i^2 / c \right]^{1/2} = dt/\gamma$$

Therefore the velocity in this 4-D space would be:

$$v_i = \frac{dx_i}{dr} = \gamma(v, ic)$$

And acceleration in this 4-D space would be:

$$a_i = \frac{dv_i}{dr} = \frac{\gamma d(\gamma(v, ic))}{dt}$$

Note is that  $t$  is that of proper time but  $\mathbf{v}$  is a 3-D in classical time.

One can define the force as well by;

$$F_i = a_i m_0 = m_0 \frac{\gamma d(\gamma(v, ic))}{dt} = \gamma \left[ \frac{F, ic m_0 d\gamma}{dt} \right]$$

However with relativistic mass the forth components of this 4-D force would be;

$$\frac{ic\gamma dm(v)}{dt}$$

For a 3-D force when  $m = m(v) = \gamma m_0$  we have;

$$F = \frac{d(mv)}{dt}$$

The momentum in this 4-D space is defined as;

$$P_i = v_i m_0 = m_0 \gamma(v, ic) = (mv, icm) = (p, icm)$$

However in 3-D space the momentum would be represented as:

$$P = mv = \gamma m_0 v$$

Thus as mentioned before for an accelerated particle:

$$m \propto v \text{ and } m \rightarrow \infty \text{ if } v = c$$

Remember that, Newton's second law holds when

$$F = \frac{dp}{dt} \text{ but does not hold when } F = \frac{mdv}{dt}.$$

To sum up all together we say since;

$$\sum_i p_i^2 = m_0^2 \gamma^2 (v^2 - c^2) = -m_0^2 c^2 \text{ and } \sum_i v_i^2 = -c^2 \text{ are constant then we have:}$$

$$\begin{aligned} 0 &= d\left(\sum v_\mu^2\right)/dr = \sum 2v_\mu dv_\mu/dr = \sum v_\mu a_\mu \\ &= (2/m_0) \sum v_\mu F_\mu = (2\gamma^2/m_0) (vF - c^2 dm/dt). \end{aligned}$$

Or more explicitly:

$$\frac{d(mc^2)}{dt} = EvF$$

Where E is the charge of energy per unit time or power. Simplifying again we get:

$$E = mc^2$$

Thus the momentum in the 4-D space would be:

$$p_i = \left( \mathbf{p}, \frac{iE}{c} \right)$$

I.e. a composite of P and E. And also;

$$\sum p_i^2 = -m_0^2$$

will become;

$$p^2 - E^2/c^2$$

Combining and taking the square root we have:

$$E = [(m_0c^2)^2 + p^2c^2]^{1/2}$$

In high velocity  $\mathbf{p}$  dominates and our famous formula will become:

$$E = pc$$

This happens for particles such as photons, neutrinos and electrons etc. however in lower velocity one can expand above square root as:

$$E = m_0c^2 + \frac{p^2}{2m_0} + \dots = E_0 + E_{kinetic}$$

Remember; in any relativistic process the mass is not constant, but both momentum and energy will be conserved.

The task of GTR is to build inertial/ non-inertial reference frames which is based on *equivalence principle*. It says; all small freely falling RFs are inertial-the first law of motion holds in the RF. A

small accelerated RF is indistinguishable from an RF, in which there exists a gravitational field. Thus this principle applies just for small i.e. local and not large or infinite i.e. global RFs. Generally, the more acceleration of the RFs the more gravity *in* them. Thus according to this principle; gravity and acceleration is equivalent to each other ( $G \equiv a$ ). *Thus every free falling object is an inertial RF.*

Gravity bends the light path since the RF is accelerated and more gravity is exerted on the beam of light. Gravity slow down the clocks; the stronger gravitational field, the slower the clocks and vice versa. This is called *gravitational time dilation*. The light (electromagnetic waves) moving in an accelerated RF (gravitational field) experiences a *Doppler Shift* i.e. EM waves are blue shifted when moving in the same direction of a gravitational field and red shifted when moving opposite it.

GTR is the combination of classical gravity and STR with a geometric feature called space-time. In fact GTR deals Geometry of gravity. According to GTR gravity is just a representation of a *curvature* of space-time i.e. the *state* and *amount* of being curved in space-time structure. The source of this gravity is the relativistic mass

in STR. According to  $E = mc^2$  the relativistic mass is equivalent with energy. Thus Einstein wrote his equation which is called *Einstein equation* as;

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi GT_{ij}$$

Where R stands for curvature, g stands for the shortest distance between two space-time points in the left-hand side. In the right-hand side we have matter (energy). Thus it joins the matter (energy) with space-time. "Matter determine the structure (curvature) of space-time and space-time determines how matter moves in the structure".

To gain more of GTR we need to know some mathematical terms such as a *tensor* field, Riemann spaces and geometry and curvature. Due to the space and complexity of the subject for an elemental book like this we guide the reader to references and further reading of this part. Now we go on with very fundamental topic of thermal motion i.e. **thermodynamics**.

Generally a thermodynamic system or body is a system composed of *intensive* (substance - independent) and *extensive* (substance-dependent) quantities to describe the state of the system. These quantities characterizes the state of the system. However, "specific extensive

quantities, i.e. the values per unit amount of substance, behave like intensive quantities".

"Thermodynamics deals with the study of the laws of thermal motion in equilibrium systems and in the transition of systems to equilibrium [states] and it also extends these laws to non-equilibrium systems".

Thermodynamic parameters describe the state of the thermodynamic system or body. They are intensive quantities such as "temperature, pressure, and specific volume (density) of the body". However another more common way of terming about these parameters is as:

"All **macroscopic attributes** characterizing a system and its relation to the surrounding bodies are called **macroscopic parameters**. They comprise, for example, such quantities as density, volume, elasticity, concentration, polarization, magnetization etc. Macroscopic parameters are sub-divided into external and internal ones".

In the case of "pure substance" i.e. when "no external forces act on the system" the state of the system determine with two intensive parameters. While in the case of a mix substance i.e. when "the system is under external forces" such as electric field or magnetic field, the number of parameters increases. Consequently, for any other parameters of state we have a function of two

given parameters. Henceforth, any three parameters of state such as pressure (P) specific volume ( $v$ ), and temperature (T) of a pure substance are uniquely and individually related to each other" in an equation called **equation of state**. We must note that;

"The set of the independent macroscopic parameters of a system determines *the state of the system*. Quantities that are independent of the previous history of the system and are fully determined by its state at a given instant (i.e. by the set of the independent parameters) are called *parameters of state*". We have **stationary state** when the parameters are constant with respect to time. This in addition to the lack of external forces is the equilibrium state of a thermodynamic system. And this is the most considering topic in thermodynamics. Therefore either macroscopic or other system in thermodynamic equilibrium are called thermodynamic system.

Reaching the **equilibrium state** is an infinitesimal and limiting process. For example in the case of density equilibrium value ( $\rho_0$ ) of a gas i.e. the number of particles per unit volume of gas, the macroscopically constant, equilibrium value expressed as an average value over a large time interval  $T$  as:

$$\rho_0 = \bar{\rho} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(t) dt$$

"Similarly, the equilibrium value of any other internal parameter is the mean value, over a long interval of time, of the function of coordinates and rates of change corresponding to this parameter".

Thus now we can define a system of coordinates of these parameters such as  $P, v, T$  as in the case of ordinary equations and treat with those equations of state similar to algebraic and differential equations. "The projections of this thermodynamic surface on the coordinate planes (say  $p$ - $v$ ,  $p$ - $T$ , or  $v$ - $T$  planes) are called **phase diagrams** of the substance.

We know that, a thermodynamic system is a collection of bodies interacting with each other and also with all other bodies beyond the boundaries of the system i.e. the surrounding medium as well. Thus any changes in at least one of the state parameters will change the state of the whole system. This is called a **thermodynamic process**. "This process is a collection of varying states of the system".

By the above discussions we can define two thermodynamic functions of *process* such as **heat**

and **work** and functions of *state* such as **Internal energy**, **enthalpy**, and **entropy** for thermodynamics quantities. The earlier deepens on the path of the process and the latter depends on "the difference of the values of the given function in the terminal and initial states".

The common formula for work is  $dw = F dt$  where  $w$  is the work  $F$  is the generalized force exerted on the body or system and  $t$  is state parameter which here represented as generalized coordinate. By these definition one can define the amount of work performs which is done by pressure  $p$  along with the volume  $V$  increase as;  $dw = p dv$ . Thus in considering a system we must always determine that, "what state parameter of the system is a generalized force and what a generalized coordinate". In the case of simultaneous exerting forces we have a summation notation of number of coordinates  $i = 1, 2, \dots, n$  as;

$$dw = \sum_{i=1}^n F_i dt_i$$

Where  $F_i$  and  $t_i$  are generalized force and coordinate. However in the case of idealized or simple system considering more than one such two exerting forces one can define the work done as;  $dw = F_1 dt + F_2 dt$ . This can be done for any

set of effecting parameters on the considering system.

Now we go on with fundament laws of thermodynamics. The first law is the law of conservation and conservation of energy which expressed as;

$$dQ = dU + dW$$

"Where  $Q$  is the amount of heat supplied to or rejected from a thermodynamic system,  $U$  the internal energy of the system, and  $W$  the work done by the system (or done on the system)".  $dQ$  and  $dU$  are total differentiations since  $Q$  and  $U$  are process (path) functions and not state functions. Thus for an idealized system with help of  $dw = p dv$  we can write,

$$dQ = dU + p dV,$$

And some extension we also can define below relation for more complex systems as;

$$dQ = dU + F_1 dt + F_2 dt$$

And or;

$$dQ = dU + \sum_{i=1}^n F_i dt_i$$

**Enthalpy** (H) is one of the very fundamental quantities in thermodynamics. Using above method we can define it as;

$$H = dU + pV$$

This way is also hold for more complex systems as.

$$H = dU + pV + FW$$

We begin the second law of thermodynamics with the expression of  $Tds \geq dQ$ . Where S is the **entropy** of the system. Note is that when  $Tds > dQ$  then the system undergoes an irreversible process and when  $Tds = dQ$  then it undergoes a reversible process.

In thermodynamics, most of the defined functions are functions of several variable therefore basically we deal with partial derivative and partial differential equations. For example in the case of rate of change (derivative) of pressure with respect to derivative of temperature (T) in at constant volume V, we write  $(\frac{\partial P}{\partial T})_v$  and also in this way for entropy S, *enthalpy* H, etc. however the total derivatives and ordinary differential equations are also practical in specific cases. For instance, in the case of inverse quantities of T and P we write;

$$\left( \frac{\partial p}{\partial T} \right)_v = 1 / \left( \frac{\partial T}{\partial p} \right)_v,$$

While in a specific case on a saturation line we express above inverse relation as;

$$\frac{dp}{dT} = 1 / \frac{dT}{dp}.$$

However the differentiation rule such as *chain rule* help us to deal with two variable cases such as  $p = p(T, v)$  and  $s = s(T, v)$  as:

$$\left( \frac{\partial p}{\partial T} \right)_v = \left( \frac{\partial p}{\partial s} \right)_v \left( \frac{\partial s}{\partial T} \right)_v$$

While in the a specific case along with boundary curve this would expressed as;

$$\frac{dp}{dT} = \frac{dp}{dv} \frac{dv}{dT}.$$

Since there is not enough space we have to finish our thermodynamic discussion here and go on with electrodynamics. However similar to all of this book, here, we tried to only consider the core idea of every subject.

At present time four elementary forces in nature have been detected i.e. "four types of interactions" of *gravitational interaction or forces*, *electromagnetic interaction or forces*, *strong*, and *weak nuclear interaction or forces*. The important note is that all other forces can be

reduce to these four forces. For now we consider core idea of electromagnetic phenomena and their interactions or forces which have an important role in nature.

We know that the gravitational forces acting between charged particles are very small in comparison with the electrical forces between them. For example "the gravitational interaction between two electrons is negligible in comparison with the electrical interaction between them". Thus, however, the basic ideas of electromagnetisms are similar to gravitation since they are both happening in a field but the latter is very stronger than the earlier. For instance, "the force of gravitational attraction between two electrons separated by a distance  $r$  is defined as;

$$F_T = G \frac{m_0^2}{r^2}$$

Where  $G$  is the gravitational constant which is equal to  $6.7 \times 10^{-11} N m^2/kg$  and  $m_0$  is the rest mass of the electron which is equal to  $9.1 \times 10^{-31} kg$ . However the electrical repulsion force acting between electrons is expressed as;

$$F_e = \frac{e^2}{4\pi\epsilon_0 r^2}$$

Here  $e = 1.6 \times 10^{19}$  coulombs (coul) is the electron charge, and  $\epsilon_0 = \frac{1}{4\pi} \times 9 \times 10^9 F/m$  is the permittivity of empty space.

Consequently combining these to relations we have a ratio as;

$$\frac{F_e}{F_T} = \frac{c^2}{4\pi\epsilon_0 G m_0^2} \approx 10^{43}$$

While **nuclear forces** are very much greater than the electromagnetic forces. They are strong interactions, in significantly short ranges or distances i.e. when "nucleons are at a distance of the order of  $10^{-13} cm$ ". Thus the greater the distance the weaker the force until it becomes negligible in comparison with electromagnetic (EM) forces. Thus these forces and interactions are existed among elementary particles to build atomic nuclei.

"Weak interactions appear in  $\beta$ -decay in which fast electrons ( $\beta$ -particles) are emitted in the course of nuclear transformations. The decay of the neutron is an example of this process.

However in the case of charged particles the only force to consider and control is the EM interactions. Thus from quantum space to GR space we are dealing with EM interactions. For instance, electrical engineering, electronics,

thermonuclear reactions, ion and plasma jet engines are problems of **electrodynamic**. "In astrophysics: the magnetic fields in interstellar space accelerate cosmic charged particles, solar flares are accompanied by considerable changes in the magnetic field close to the sun's surface, and the earth's magnetic field confines charged particles in the neighborhood of the earth, thus creating radiation belts". Consequently, the theory and laws of EM interactions or electrodynamics describe fundamental laws of nature. However we must note that, "not all elementary particles are sources of electromagnetic field. Particles that create such a field are ascribed a definite value of the electric charge ( $e$ )".

To begin with the electromagnetism and electrodynamics is better to start with their very basic concepts of **charge** and **field** in a vacuum. Hence we are dealing with vacuum Electrodynamics. This field of force is caused or created by the interaction among neighboring charges. Thus 'interaction of particles can be described with the help of the concept of a *field* of force".

Suppose a particle moving in an electromagnetic field, then we have two actions to consider i.e. the action for the free particle and the actions on or

interactions between the particle and the field it is moving in. For interactions we need quantities to characterize the particle and also quantities to characterize the field.

The charge is a quantity for a particle similar to the mass. The value of  $e$  "does not depend on the choice of the reference frame".

However there exist negative and positive particles, but the laws are symmetrical from  $e^+ \rightarrow e^-$  i.e. the conservation of electric charges. This is also true for a closed system of particles. Thus one can write this as;

$$\sum_i e_i^+ - \sum_i e_i^- = \text{constant}$$

Consequently; "the electric charge of a particle is one of its basic characteristics. It has the following fundamental properties:

- (1) Electric charge exists in two forms, i.e. it can be positive or negative;
- (2) The algebraic sum of charges in any electrically insulated system does not change i.e. *the law of conservation of electric charge is hold.*
- (3) Electric charge is a relativistic invariant: its magnitude does not depend on the reference system, in other words, it does not depend on whether the charge moves or is fixed".

If  $q$  is a charge creating an electric field and one put another fixed point charge  $q'$  on this field with intensity  $E$  at the given point charge then  $q'$  experience a force  $F$  as;

$$F = q'E$$

Here vector  $E$  is defined as the force acting on a positive fixed unit charge and  $q' < q$ .

From experiments it is showed that, according to coulomb's law, the value of  $E$  at a distance  $r$  from it could be expressed as;

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} e_r$$

This formula express *Coulomb's law* in a form of a field. Where  $\epsilon_0$  is the electric constant in the inverse ratio with the value of;

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ m/F}.$$

In fact similar to the case of invers square law of universal gravity, here also the intensity  $E$  of the field created by a point charge is inversely proportional to the square of the distance  $r$ . Here  $e_r$  is the unit vector with radius  $r$  from the center of the field. "The charge  $q$  is measured in *coulombs* C and the field intensity  $E$  in *volts per metre* V/m. Vector  $E$  is directed along  $r$  or oppositely to it, depending on the sign of the charge  $q$ ".

However the intensity of the vector  $E$  for a system of fixed point charges is expressed as;

$$E = \sum E_i = \frac{1}{4\pi\epsilon_0} \sum \frac{q_i}{r_i^2} e_{ri}$$

This statement is called the **principle of superposition**. However it is better to consider a system of particles a continuous distribution. This would able us to use integrations to introduce, charge density as volume density  $\rho$ , surface density  $\sigma$ , and linear density  $\lambda$  as below;

$$\rho = \frac{dq}{dV}, \quad \sigma = \frac{dq}{dS}, \quad \lambda = \frac{dq}{dl},$$

"Where  $dq$  is the charge contained in the volume  $dV$ , on the surface  $dS$ , and in the length  $dl$  respectively". Thus our summation formula above can be written as an integration over the distribution space with nonzero values of  $\rho$ ;

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \mathbf{e} dV}{r^2} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \mathbf{r} dV}{r^3}$$

The note is that as  $E$  is a vector thus for calculating it we need to perform this integration for each of components.

The geometrical representation of any field usually is based something called field lines. In the case of vector  $E$ , if we know its values at each point, then, electric field can be visually

represented with the help of field lines. The resulted pattern shows the direction and magnitude of vector  $E$  i.e. its configuration.

Therefore the field  $E$  has two important very handy properties. The Gauss theorem and the theorem on circulation of vector  $E$ , which are associated with the idea of *flux* and *circulation*. "In terms of these two concepts not only all the laws of electricity but also all the laws of magnetism can be described".

As we know, the *divergence* and *rotation* can also be given invariant definitions: the former on the basis of the motion of a flux and the latter on the basis of circulation". In vector analysis the divergence help us to have **Gauss formula** for the **flux** of a vector or more precisely of a vector field through a closed surface  $\sigma$  which is defined as the integral;

$$\oint_{\sigma} A \cdot d\sigma = \oint_{\sigma} A_n d\sigma$$

Then as;

$$\oint_{\sigma} A \cdot d\sigma = \int_{\Omega} \text{div } A d\Omega$$

Where  $\Omega$  the domain or region is bounded by the surface  $\sigma$  and  $A_n$  is the projection of  $A$  on the outer normal to  $\sigma$  which goes in the direction

from interior of  $\Omega$  the surrounding space. The rotation enters into another important formula known as **Stokes' formula** which expresses the **circulation** of a vector field over a closed oriented contour  $L$  the circulation is defined as the integral

$$\oint_L A \cdot dr = \oint_L A_\tau dL$$

And expressed in terms of a surface integral by the formula

$$\oint_L A \cdot dr = \int_S \text{rot } A \cdot ds$$

Where  $r$  is the radius vector,  $\tau$  is the unit tangent vector to  $L$  in the direction of traversing the contour and  $S$  is an arbitrary surface bounded by the contour  $L$  and coherently oriented with  $L$ . however these formulas are simpler for the plane fields.

To summarize and concisely add to above discussions we say;

- An electric charge at fixed point gives rise to an *electric field* **E**.
- A moving charge causes *magnetic field* **B**.
- **E** and **B** have the same unit since in fact they are relativistic components of an

antisymmetric  $4 \times 4$  field tensor. "The electromagnetic tensor or electromagnetic field tensor is a mathematical object that describes the electromagnetic field in space-time".

- An oscillating charge causes electromagnetic waves.
- Basic form of Electromagnetic force in  $F = q(E + (v/c) \times B)$  on an electric charge moving with velocity  $v$  where  $c$  is the velocity of light.
- If  $q_1$  and  $q_2$  be two stationary (at rest) electric charges at a distance  $r$  from each other then there exist an central force  $F$ , behaves as  $\frac{1}{r^2}$  in coulomb's law as;

$$F = C \cdot \frac{q_1 q_2 e_r}{r^2}$$

Where  $C$  and  $e_r$  are the constant and unit vector mentioned before.

- This  $F$  is an Isotropic force i.e. with the same value or magnitude in every directions.
- The *strength* or *intensity* of the field  $E$  is defined by the force on a positive unit charge  $q$  as  $F = q E$ . then coulomb's law for the field about a point charge  $q$  become;

$$E = \frac{q}{r^2} e_r$$

- Coulomb forces are conservative.

Due to the last point mentioned above and also because of the basic similarity between fundamental definitions of masses and charges we continue with potential energy of a unit charge also called **potential** ( $\varphi$ ) and write;

$$E = -grad \varphi = -\nabla \varphi$$

We can also define the potential difference as;

$$U = \varphi(q_2) - \varphi(q_1) \text{ volt } V$$

Consequently the Coulomb potential for a charge  $q$  is;  $\varphi = q/r$ .

Now the *superposition principle* we mentioned before becomes linear as; the fields  $E$  arising from different point charges.

$$E = \sum_i \frac{q_i e_i}{r_i^2} = \int \left( \frac{\rho(r) e_r}{r^2} \right) d^3 r$$

At the origin of the associated coordinate for field, the potential would be;

$$\varphi = \int \frac{\rho(r)}{r} d^3 r$$

For an arbitrary position this relation would be;

$$\varphi(r) = \int \frac{\rho(r')}{|r - r'|} d^3 r'$$

Sometimes above formulations are called **electromotive force (voltage)**. The most common definition of EMF is based on a circuit. "This is the work done by the forces of an electric field in the passage of a unit charge along a given circuit".

The force acting on a unit charge at a given point is, by definition, the electric field  $E$ . The work done by this force on an element of path  $dl$  is the scalar product  $E dl$ . Then the work done on the whole closed circuit, or the EMF, is equal to the integral,

$$W = \int E dl$$

Suppose a surface is stretched over a given circuit. If a magnetic field  $B$  existed, then the magnetic flux across an element of the surface is,  $d\Phi = B ds$ . Thus for the whole surface;

$$\Phi = \int B ds$$

The amount of  $\Phi$  does not depend on the specific form of the surface stretched on the circuit since the magnetic field lines cannot originate or terminate in an empty space devoid of magnets. Therefore, if two different surfaces are stretched over the circuit, the flux across each must be the same i.e. isotropic. With this it follows the so called **Faraday's induction law** as;

$$W = -\frac{1}{c} \frac{\partial \Phi}{\partial t}$$

Where  $c$  is the velocity of light and in the case of a circuit in vacuum,  $W$  increases its energy.

Now let finish this discussion with Maxwell's equations. Maxwell himself said;

"The theory I propose may therefore be called a theory of the Electromagnetic Field, because it has to do with the space in the neighborhood of the electric or magnetic bodies, and it may be called a Dynamical Theory, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced. The electromagnetic field is that part of space which contains and surrounds bodies in electric or magnetic conditions."

We mentioned to them before in this book. Now we consider them here since they gathered all the core idea of electromagnetism in a set of differential equations. Thus using three previous relations we can define;

**Maxwell's Equation for curl  $E$  as;**

$$\int E \, dl = -\frac{1}{c} \frac{\partial}{\partial t} \int B \, ds$$

The LHS, of the equation can be transformed by Stokes' theorem, and in the RHS, the order of

time differentiation and surface integration can be interchanged, since they are performed for independent variables. Now doing integration in the LHS we get below;

$$\int (\text{curl } E + \frac{1}{c} \frac{\partial B}{\partial t}) ds = 0$$

This is hold if differential form of *faraday's induction law* for "infinitely small circuit" hold to zero i.e.

$$\left( \text{curl } E + \frac{1}{c} \frac{\partial B}{\partial t} \right) = 0$$

Since the initial circuit can have arbitrary magnitude and shape.

**Maxwell Equation for div  $B$  as;**

$$\text{div } B = 0$$

Why? Since;

$$\int B ds = 0$$

Using *Gauss theorem* we get volume integral of this as;

$$\int \text{div } B dV = 0$$

Now for an infinitely small volume we get the differential form of  $\int B ds = 0$  as;  $\text{div } B = 0$ .

Therefore we have a vanishing divergence of magnetic field  $B$  everywhere in free space due to the fact that, "magnetic lines of force cannot originate or terminate in vacuum, that is, they are either closed or go off to infinity".

However "the divergence of a vector is the density of sources of a vector field" such as in  $E$  but this is not hold in the case of  $B$  since "a magnetic field does not correspond to any free charges".

To add up until here we have got;

$$\left( \text{curl } E + \frac{1}{c} \frac{\partial B}{\partial t} \right) = 0$$

And

$$\text{div } B = 0$$

As the first pair of the Maxwell's equations. Now we go to the second pair.

**Maxwell's Equation for  $\text{div } E$ .** based on coulomb's law for the field about a point charge  $q$ ,  $E = q/r^2 e_r$  and according to Gauss law we get;

$$\int E \, ds = 4\pi q$$

If there is an arbitrary charge distribution inside a closed surface. This means that, "the electric flux across a closed surface is equal to the total electric

charge inside the surface ( $q$ ) multiplied by  $4\pi$ ".  
Therefore

Now if we rewrite  $E = q/r^2 e_r$  as;

$$E = \frac{q}{r^2} \frac{r_e}{r}$$

Now again, the field  $E$  is inversely proportional to  $r^2$  and is directed along the radius vector  $e_r$ .

For a charge with surrounding spherical surface and solid angle  $\Omega$  we have,

$$r^2 d\Omega \frac{r_e}{r}$$

Here  $\frac{r_e}{r}$  specifies the direction of the normal to the surface. Thus here the flux of  $E$  is;

$$E ds = \frac{q}{r^2} \times \frac{r_e}{r} \times r^2 d\Omega \frac{r_e}{r} = q d\Omega$$

For the whole surface we write this relation as;

$$\int q d\Omega = q \int d\Omega = 4\pi q$$

Consequently from above relations, we say, the flux through any closed surface will be the same due to the fact that, lines of force begin only at a charge.

Now, if  $q$  is a function of  $\rho$  then;  $q = \int \rho dV$  and density of point charges be defined as a limiting process as  $\rho = \lim_{\Delta V \rightarrow 0} \Delta q / \Delta V$ , we have, the differential form of the flux of  $E$  as;

$$\int (\text{div } E - 4\pi\rho) dV = 0$$

Thus for an arbitrary infinitely small volume we get;

$$\text{div } E = 4\pi\rho$$

This means that, "the density of sources of an electric field is equal to the electric charge density  $\rho$  multiplied by  $4\pi$ ".

Note is that for finite charge and volume the density of point charges is;  $\rho = q/\Delta v$  but when the volume is infinitely small i.e. tends to zero, the density of point charges  $\rho$  becomes a function which is equal to zero everywhere except at the point of the charge.

Now we go on with the charge conservation law which we mentioned before in a more details and displacement current to get the last Maxwell's equation.

In a system with no external charges coming in, the total charge of the system remains unchanged. Thus the current of charges existed and they have densities. In fact there is a quantity called **current density**  $J$ . this is a vector quantity for charges per  $\text{cm}^2$  per second expressed as;

$$J = \rho v$$

Where  $v$  is the charge velocity at the point where the density (function)  $\rho$  is defined. The total current  $I$  emerging from an area is defined as;

$$I = \int J ds$$

Now with the charge conservation law,  $I$  must be equal to the decrease of charge inside the surface in unit time. this is expressed as;

$$I = -\frac{\partial q}{\partial t}$$

If we put  $q = \int \rho dV$  and transform  $I$  by the Gauss theorem, then the charge conservation law in arbitrary volume  $V$  would express as;

$$\int \left( \frac{\partial \rho}{\partial t} + \text{div } \rho v \right) dV = 0$$

Consequently the charge-conservation law in differential form is;

$$\frac{\partial \rho}{\partial t} + \text{div } J = \frac{\partial \rho}{\partial t} + \text{div } \rho v = 0$$

We know from direct and alternating currents theories that, direct-current vector lines are always closed while "open lines indicate that there is either an accumulation or loss of charge at their ends". On the other hand in the case of alternating currents. the vector lines will always be closed or go to infinity.

Therefore replacing  $\text{div } E = 4\pi\rho$  instead of  $\frac{\partial\rho}{\partial t}$ , we get;

$$\text{div} \left( J + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right) = 0$$

Here  $J + \frac{1}{4\pi} \frac{\partial E}{\partial t}$  are the closed vector lines and  $\frac{1}{4\pi} \frac{\partial E}{\partial t}$  is the displacement current since it is independent of charge transfer. Thus we have a system with closed vector lines.

### **Maxwell's Equation for curl B.**

For this equation we need, law of Biot-Savart and magnetic action of the displacement current. The earlier says at a point with radius vector  $r$  from  $r_1$  there is a linear current produces a field  $dB$  defined as:

$$dB = \frac{I dl_1(r - r_1)}{c |r - r_1|^3}$$

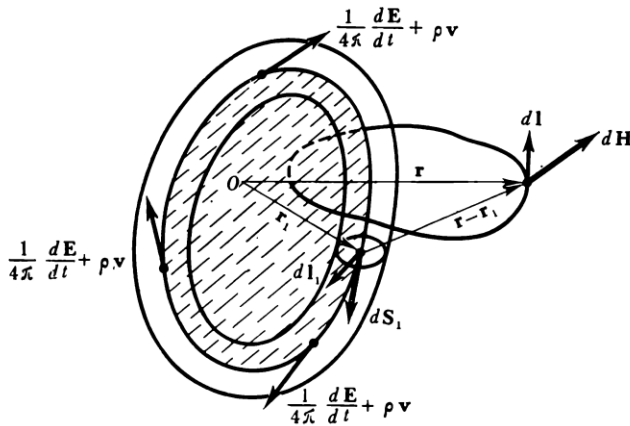
Here  $I$  is current strength,  $dl_1$  length element of current. In the case of a spacially distributed current we have the intensity as;  $I = \rho(v ds)$ .

Replacing this in above relation we have;

$$dB = \int \rho(v ds) \int \frac{dl_1(r - r_1)}{c|r - r_1|^3}$$

Adding displacement current we get;

$$\int B dl = \frac{1}{c} \int \left( \rho v + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right) ds \int dl \int \frac{dl_1(r - r_1)}{c|r - r_1|^3}$$



According to figure above "if the hatched contour and the vector line contour along which integration over  $d\mathbf{l}$  is performed are connected" then the area  $A$  is;

$$A = \int d\mathbf{l} \int \frac{d\mathbf{l}_1 (\mathbf{r} - \mathbf{r}_1)}{c |\mathbf{r} - \mathbf{r}_1|^3} = 4\pi$$

If  $\mathbf{r}_1 = \nabla_1$  and  $\mathbf{r} = \nabla$  then  $\nabla_1 = -\nabla$  since the integrand depend on  $\mathbf{r} - \mathbf{r}_1$ . Thus

$$\frac{(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} = \nabla_1 \frac{1}{|\mathbf{r} - \mathbf{r}_1|}$$

Thus we have;

$$\begin{aligned} A &= \int d\mathbf{l} \int \left( d\mathbf{l}_1 \times \nabla_1 \frac{1}{|\mathbf{r} - \mathbf{r}_1|} \right) \\ &= \iint d\mathbf{l}_1 \left( \nabla_1 \frac{1}{|\mathbf{r} - \mathbf{r}_1|} \times d\mathbf{l} \right) \end{aligned}$$

Applying Stokes' theorem we get;

$$A = \iint \text{curl}_1 \left( \nabla_1 \frac{1}{|r - r_1|} \times dl \right) dS_1$$

Now using vector product rule and since

$$\text{div} \nabla_1 \frac{1}{|r - r_1|} = \nabla_1^2 \frac{1}{|r - r_1|}$$

Then;

$$\Omega = - \int dl \nabla \int \frac{(r - r_1) dS_1}{|r - r_1|^3}$$

Here  $\Omega$  is the *total solid angle* at which the path described by the current line in the figure is seen from the point on the circuit  $dl$ . Thus  $dl \nabla \equiv d\Omega$  along the circuit  $l$  and to calculate its integral of  $4\pi$  of  $\Omega$  variation i.e.  $2\pi - (-4\pi)$ . Consequently;

$$\int B dl = \frac{4\pi}{c} \int \left( \rho v + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right) dS$$

Finally we get maxwell's fourth equation using stock theorm as;

$$\text{curl } \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

To summerize all four equations we can express them as below in integral form and words as below;

$$\text{Gauss' law: } \iint \mathbf{E} \cdot d\mathbf{S} = Q ,$$

Electric flux  $E$  out of any closed surface  $dS$  = Charge inside  $Q$ . this is Equivalent to Coulomb's law.

No magnetic monopole:  $\iint \mathbf{B} \cdot d\mathbf{S} = 0,$

Magnetic charge does not exist.

Electromagnetic induction:

$$\oint \mathbf{E} \cdot d\mathbf{x} = -\frac{1}{c} \frac{d\Phi}{dt},$$

Energy gained by test/point charge traversing any closed circuit is proportional to rate of change of magnetic flux through circuit.

Ampere's law:  $\oint \mathbf{B} \cdot d\mathbf{x} = \frac{1}{c} I.$

Current generates magnetic field running in rings around the current.

*\*dx is the same as dl.*

Here the last equation does not hold for charge conservation law, in the case of time varying field. To deal with this problem we expressed them in differential along with the integral forms. We did this based on introducing the idea of infinitesimal or "infinitely small" surfaces, circuits and volumes i.e. their limits tend to zero. And for the last equation by introducing displacement current i.e.  $\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.$

The result was a set of Maxwell's differential equations forming, foundations of electromagnetism and electrodynamics as;

$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B},$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E}.$$

In fact  $\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$  make it possible to have electromagnetic wave equation since this set of 8 equations "imply that a disturbance in the electromagnetic field will propagate at velocity  $c$  as a constant, charge distribution (density) in space  $\rho$  and current distribution (density) in space  $\mathbf{j}$  as known quantities. While field parameters  $\mathbf{E}$  and  $\mathbf{B}$  are unknowns vectors with three components in  $x, y, z$  directions. However we must note that, with respect to the number of components in every fields  $\mathbf{E}$  and  $\mathbf{B}$ , only six of them are independent i.e. 3 components of  $\mathbf{E}$  and 3 of  $\mathbf{B}$ . This is because, "the three components of each rotation are constrained by  $\text{div rot} = 0$ .

Another unknown quantities are **electromagnetic potentials** which we mentioned earlier. With these quantities, each equation just have one unknown. This leads to the reduction in number of equations.

If  $A$  be a vector potential and  $\varphi$  a scalar potential then we have;

$$B = \nabla \times A \equiv B = \text{rot } A$$

This means, adding "the gradient of any arbitrary 4-D function  $f(x, y, z, t)$  to the vector potential, the magnetic field will not change, since the rotation of a gradient is identically equal to zero". This also can be expressed as;

$$A = A' + \nabla f(x, y, z, t)$$

To result that;  $B = \text{rot } A = \text{rot } A'$

And thus by using  $\text{rot } B = -1/c \frac{\partial B}{\partial t}$  for scalar potential we get;

$$E = -\nabla \varphi - \frac{1}{c} \frac{\partial A}{\partial t}$$

And being invariant is also hold for electric field with addition of the gradient of the 4-D function i.e.

$$\varphi = \varphi' - \frac{1}{c} \frac{\partial f}{\partial t}$$

Consequently with these arbitrary potentials, we can get the fields as well. These 4 potentials hold "all the information about the 6 field components" in addition to a bunch of others. This is a key property, called **gauge invariance** and those two

transformation relations are called **gauge transformations**.

Now applying  $A$ ,  $\varphi$  and  $f$  to;

$$\text{rot } B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi J}{c}$$

We get Lorentz condition as;

$$\text{div } A' + \frac{1}{c} \frac{\partial \varphi'}{\partial t} = 0$$

Thus "the expression of fields in terms of potentials is not changed by a gauge transformation". While this transformation is performed under Lorentz condition.

Now we consider *principle of least action* for electromagnetic field. This is due to the fact that the laws of electrodynamic are analogous to the laws of mechanics. Thus here we are dealing with principle of variation and calculus of variation. However we will not go to the details and just clarify the core ideas.

In our discussion about STR and GTR we explained that how from classical relativity principle we get general Lorentz transformations, based on the Einstein relativity principle. We also discussed the concept of generalized coordinate, Lagrangian and Hamiltonian based on the principle of least action. Now we apply them in the same way but with a different form to adjust with the electromagnetic field equations. The

main points is that in electromagnetic field the number of degrees of freedom is infinite and the parameters are continuously varying. Thus the integration come along, over volume occupied by the field using potentials  $A(r, t)$  and  $\varphi(r, t)$  as the generalized coordinates  $q_i(t)$ . The value of  $r$  in here corresponds to the number of generalized coordinates ( $i$ ). Thus the lagrangian form would be as;

$$L = \int \left( \frac{E^2 - B^2}{8\pi} + \frac{AJ}{c} - \varphi\rho \right) dV$$

Where  $E^2 = \left(\frac{1}{c}\frac{\partial A}{\partial t} + \nabla\varphi\right)^2$  and  $B^2 = (\text{rot } A)^2$  are equations of potentials. Note is that the with respect to gauge invariance the invariance of action will be satisfied as well.

Using above formulations and the lagrangian relation  $E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$  one can get the energy of electromagnetic field without charges i.e. the usual mechanical energy as;

$$E = \frac{1}{8\pi} \int (E^2 + B^2) dV$$

However we can also show that in the presence of charges the total energy is conserved. To do this we need to do two scalar product of

$$\text{rot } E = -\frac{1}{c} \frac{\partial B}{\partial t} \text{ and } \text{rot } B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi J}{c}$$

By E and B as

$$\frac{1}{c} \left( E \frac{\partial E}{\partial t} + B \frac{\partial B}{\partial t} \right) = E \operatorname{rot} B - B \operatorname{rot} E - \frac{4\pi J E}{c}$$

Now since divergence of a vector product of vectors A and B is;

$$\operatorname{div} [AB] = B(\nabla_A A) - A(\nabla_B B) = B \operatorname{rot} A - A \operatorname{rot} B$$

We get;

$$\frac{\partial}{\partial t} \left( \frac{E^2 - B^2}{8\pi} \right) = -\operatorname{div} \frac{1}{c} 4\pi [EB] - J E$$

since  $J = \rho v$  we get;

$$\begin{aligned} \frac{\partial}{\partial t} \int \left( \frac{E^2 - B^2}{8\pi} \right) dV \\ = - \int \frac{c}{4\pi} [EB] ds - \int \rho v E dV \end{aligned}$$

And finally if K be equal to kinetic energy of the charge in unit time then;

$$\frac{d}{dt} (E + K) = - \int \frac{c}{4\pi} [EB] dV$$

This is equivalent to the work done in unit time.

"Thus, the decrease in energy, in unit time, of an electromagnetic field and of the charged particles contained therein is equal to the vector flux  $\frac{c}{4\pi} [EB]$  across the surface bounding the field. If this surface is infinitely distant and the field on it is equal to zero, what we have is simply the

energy conservation law for an electromagnetic field and for the charges within it". While, if in a finite volume the integral of:

$$- \int \frac{c}{4\pi} [EB] dV$$

Represent the energy passes in unit time through the surface bounding the volume. Thus the energy crossing unit area in unit time or, more simply, the energy density flux vector or the Poynting vector is;

$$U = \frac{c}{4\pi} [EB]$$

In the same way the linear momentum of an electromagnetic field represent as;

$$P = \int \frac{c}{4\pi} [EB] dV$$

Where  $\frac{c}{4\pi} [EB]$  is the field momentum density.

Thus if  $r$  be the radius vector then;  $\frac{c}{4\pi} [r[EB]]$  is the angular momentum density. Consequently the total angular momentum of an electromagnetic field is;

$$M = \int \frac{c}{4\pi} [r[EB]] dV$$

The conservation law of charges in an electromagnetic field is also hold for these two momentums. However in our next discussion

i.e. quantum mechanics, angular momentum is very crucial.

According to above similar to Hamiltonian function

$$\frac{p^2}{2m} + K \frac{x^2}{2}$$

Of the harmonic oscillator leads to an energy transfer between kinetic and potential energies. The energy density i.e.  $E^2 + B^2$  make it possible to have electromagnetic waves. Obviously in an electromagnetic wave the electric and magnetic energies exchange. Remember the momentum density  $\frac{c}{4\pi} [EB]/c^2$  will be unchanged in the wave field.

Until here we introduce two sorts of motion in nature. The motion along the trajectory and the wave motion which exist in the case of sound and light." light consists of electromagnetic oscillations, i.e., periodic variations in the electromagnetic field in space and in time". However one of the first scientists that consider the light as a stream of particles was Isaac Newton. He named this stream as *corpuscles*. In fact, by particle concept we have **geometric optics**. Basically this subject is about finding relations between the wave and particle nature of

light i.e. "the trajectories of the corpuscles correspond to rays of light".

For instance, light motion in a homogeneous medium is a straight line trajectory and also the trajectory of a particle in the absence of acting forces is a straight line. The law of reflection is applicable for both. In earlier by a mirror and in the latter by a barrier.

Therefore, until here, we have displacement of bodies along trajectories and propagation of waves. But the point is that, the propagation of wave cannot always be reduced to the motion/displacement of particles in space. For example in the case of diffraction rings around a lenses of a good telescope which are connected to the wave nature of light.

On the other hand, the laws of electron optics, deduced either from the corpuscular theory, or from Huygens' wave laws. In large regions, the corpuscular and the wave approaches have the same results, but in small regions conditions change. The question is how small?

We know that, if a beam of electrons is passed through a crystal, the resulting diffraction pattern is the same as that of X rays which are Electromagnetic waves with much smaller wavelengths than light. Thus the point is the **diffraction of electrons**. They are not waves but

particles. However some experiments showed the wave nature of electrons. Therefore now, the question is about this paradoxical behavior.

Light shows this paradoxical behavior as well. "When refracted by a lens, it travels in a straight line, but when it strikes a diffraction grating it exhibits its wave nature". In fact it depends on "the ratio between the wavelength and the size of the region in which the motion takes place". Thus what wavelength is associated to an electron?

Before answering this question let's consider some fundamental facts about wave and wavelength which has different definitions such as;

In the case of a periodic wave in an isotropic medium, the wavelength is the "perpendicular distance between two equal phase surfaces in which the displacements have a difference in phase of one complete period". Another definition says the wavelength is "the space period of a wave, i.e., the least translation distance that leaves the wave invariant". In the case of a harmonic wave "the wavelength is the distance between any two points at which the phase at the same instant differs by  $2\pi$ , specifically if the wave is a plane harmonic wave in the x-direction with angular frequency  $\omega$ , so that the disturbance has the form of;

$$\sin (\omega t - kx)$$

then the wavelength is

$$\lambda = \frac{2\pi}{k} = 2\pi v/\omega$$

However in a related point we could mention that, in group velocity (G) of waves which is similar to the velocity of energy propagation, a bunch of waves travelling with a much closed frequency in a medium in which the phase velocity is a function of frequency expressed as;

$$G = c - \lambda \frac{dc}{d\lambda}$$

Here c is the wave velocity at any  $\lambda$ .

Now to answer to our question we say;

According to the basic relation of quantum mechanics the energy of a photon is equal to a multiplication of Planck constant  $h$  by frequency  $\omega$  i.e.

$$E = h \omega$$

This together with "the energy which an electron receives in falling through one **volt** of **potential difference**", we get  $\lambda$  associated with one electron-volt of;

$$12397.67 \pm 0.22 \times 10^{-8} \text{ cm}$$

This is the wavelength ( $\lambda$ ), corresponds to the motion of an electron. For more details and illustration we consider X-ray. In the case of X rays the equation in use is,

$$2d \sin \theta = n\lambda.$$

Where  $d$  is the distance between the planes and the path difference is  $2d \sin \theta$ . This is equal to  $n\lambda$  where  $n$  is an integer. Using this equation,  $\lambda$  could be measured "directly from the diffraction pattern". One can do this for other particles such as protons by scattering them on crystal surface instead of transmitting through and determine  $\lambda$  as well. The result is;  $\lambda$  "is inversely proportional to the momentum ( $p = mv$ ) of the particle".

Until here we get that, a particle in our discussion, has two important properties i.e. the wavelength and the momentum. Thus there should exist a coefficient of proportionality between them ( $\lambda$  and  $mv$ ) which is a universal constant  $h$ . it is called **Planck's constant**. Thus, *Planck's constant* ( $h$ ) is "the coefficient of proportionality between the wavelength and the momentum of the particle". Consequently we get famous relation of;

$$\lambda = h/mv$$

In fact this comes from the work of Max Planck on "thermodynamics concerning the relationship between energy and wavelength". Thus the unit of this constant expressed in two ways. Firstly since another form of above formula, is,  $h = \lambda mv$  then  $h$  has the dimensions of  $\lambda mv$  i.e.

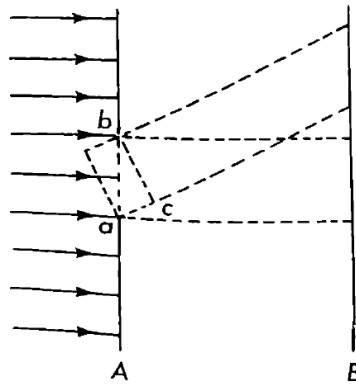
$g\text{ cm}^2/\text{sec}$  in CGS system and these are also the dimensions of energy in time. However, energy has the dimensions of  $g\text{ cm}^2/\text{sec}^2$ .

However if we take the mass of an electron as the unit of mass and its charge as the unit of charge, then by adding Planck's constant to our list, instead of metric system, we get atomic system of units. The unit of length in this system is  $0.52917 \times 10^{-8}\text{ cm}$ , as the linear dimensions of an atom. And In the CGS system of units,  $h = 6.625 \times 10^{-27}$ . In the same way the energy  $E$  of an electron ( $e$ ) moving through a potential difference ( $\phi$ ) of one volt increases by one electron-volt (1eV). The voltage  $V$  is the potential difference ( $\phi$ ) in which the electrons are accelerated. Thus as the electron passes through the potential difference ( $\phi$ ) it obtains an energy equal to  $(e\phi)$ . The mass of an electron is equal to;  $9.107 \times 10^{-28}g$ . "The charge of an electron equals  $4.8026 \times 10^{-10}$  electrostatic units and a volt equals  $1/300$  of an electrostatic unit. Hence one electron-volt equals  $1.6 \times 10^{-12}$  ergs".

Light, is a wave motion, cannot diverge too far from the direction of the its ray. Therefore, particle trajectory exist, only, when **amplitude** of the wave motion rapidly tend to zero.

Most books from this point on, consider the famous system of an electron gun or wave source and slits. But we prefer to simply say; in considering a restriction on the propagation of waves, "only an infinitely wide beam of rays can be strictly parallel, but then it has no lateral boundary. Only such a beam corresponds to a perfectly plane wave front, which gives a single, precisely defined direction of wave propagation. The momentum of the particle coincides with this direction, and hence a particle for which the momentum is defined precisely with respect to magnitude and direction has no defined position, i.e., the beam in this case is *infinitely wide*, with its trajectory extending through all space". Thus, the velocity of the particle undergoes a certain deviation in the plane of restriction with an *uncertain* angle of diffraction.

Therefor there exist an uncertainty in the position of the particle ( $\Delta x$ ) and also in its velocity ( $\Delta v$ ). With this we can define the uncertainty in the momentum of the particle as;  $\Delta p = m\Delta v$ . Now using,  $\lambda = h/mv$ , and the figure below;



We write;

$$\Delta v \approx \frac{2(ac)v}{ab} = \frac{v\lambda}{d} = \frac{v}{d} \left( \frac{h}{mv} \right) = \frac{h}{dm}$$

The number 2 is due to the up and down deviations. And finally we get the famous **uncertainty principle** as;

$$\Delta p \Delta x \approx h$$

However one can show more precisely that;

$\Delta p \Delta x \geq \frac{1}{2} h / 2\pi$ . "The concept of an electron trajectory has reasonable meaning if the uncertainties of all three momentum components  $\Delta p_x$ ,  $\Delta p_y$ ,  $\Delta p_z$  are small compared with the momentum itself" i.e.

$$\Delta p_x \ll p_x, \Delta p_y \ll p_y, \Delta p_z \ll p_z$$

The same can be applied to other elementary particles such as a proton, neutron, meson and the like.

We know that, "the smaller the kinetic energy, the smaller the velocity of the particle". We also know that, wavelength is inversely proportional to velocity, refraction varies from point to point in an infinitely wide scope and kinetic energy  $K$  is the difference between, total energy  $E$  and the potential energy  $U$ , i.e.

$$K = E - U$$

Since in another definition  $K = \frac{1}{2}mv^2$ , we get velocity as;

$$v = [2(E - U)/m]^{1/2}$$

And wave length as;

$$\lambda = \frac{h}{mv} = h/[2m(E - U)/m]^{1/2}$$

We know that, we can plot the behavior of a variable with respect to other variable or variables i.e. a *function* or *mapping*. This is also can be done be done in the case of wave function behavior with respect to potential energy curve. In our discussion one can find, the trajectory of a particle from the behavior of the potential energy. But due to the infinitely wide scope of the

trajectories we are dealing with the **probability amplitudes**.

This is called in physics, **quantum mechanics** which is as have seen a continuation of classical mechanics.

Rutherford's experiments in 1910 showed that an atom consists of a heavy positive nucleus at the center with dimensions very smaller than the whole atom along with negative orbiting electrons round the center. His model was insufficient since "electrons will experience centripetal acceleration and a charged particle undergoing acceleration radiates electromagnetic waves, thereby transmitting its energy to the electromagnetic field. Thus, the energy of an electron moving around a nucleus should continuously diminish until the electron falls onto the nucleus" and the whole structure of the atom will collapse. In 1913, N. Bohr proposed that, the radiation is not due to the orbiting motion of electrons round the nucleus but because of "making a transition from an orbit of higher energy to one with lower energy". Thus we need differences of energy levels to have a stable atom and radiating electrons. Therefore the frequency of this radiation, such as in two levels can be expressed as;

$$h\omega = E_1 - E_2$$

Where  $h$  is a universal constant equal to  $1.054 \times 10^{-27}$  erg-sec.

While the energy balance of photoelectric effect in quantum mechanics expressed as;

$$E = h\nu$$

The experiments and discussions on diffraction patterns of electrons, Louis de Broglie proposed the wave properties motion of electrons then Schrödinger proposed the wave function. Parallel to him Heisenberg put forward the uncertainty principle. He showed that "in quantum mechanics one has only a certain definite **probability** for the occurrence of a particle or other event". This is all done by experiments along with the mathematics. Thus we get to the point that, to explain the pattern of diffraction of electrons we express this behavior with a (complex) wave function whose phase determines the diffraction pattern. Since Electrons move independently, consequently, "the number of electrons in an element of volume  $dr$  is proportional to the probability of the appearance of one electron". This real square probability is similar to wave intensity and defined as;

$$d\phi = |\psi(x, y, z, t)|^2 dr$$

Here  $d\phi$  is the probability of finding an electron in the volume  $dr$  at the instant  $t$ ; then  $|\psi|^2$  is the probability referred to unit volume or, otherwise, the **probability density**.

Let's go on with more rigorous way. According to our discussions, we can propose that probability of finding a particle at the position  $\mathbf{r}$  at the time  $t$  using wave equation as;

$$\Psi = \Psi(\mathbf{r}, t)$$

here  $\Psi$  is a complex **wave function**. For example, an electron behaves like a wave if;

$$\Psi \approx e^{iQr}$$

It behaves like a particle if;

$$\Psi = \delta(\mathbf{r})$$

I.e. a **delta function**. Or it behaves as a function between these two.

A function  $\Psi(\mathbf{r}, t)$  or  $\Psi(\mathbf{x}, t)$  is a function of radius vector components or coordinates and time. This function determine the state of a system and satisfy the so-called Schrödinger equation expressed as;

$$H\Psi = i\hbar\Psi/dt$$

Here  $H$  is the Hamiltonian operator and  $i = (-1)^{1/2}$ . in fact a wave function is a solution of an ordinary or partial differential equation for wave propagation through a particular medium.

Now we tend to consider Schrödinger equation in more details since it is the most important basic blocks of quantum theory. As we said in the part of the physic story, in 1926 Schrödinger developed a general wave equation for phase waves which was postulated by De Broglie three years before. If  $k$  be a 3-D wave operator, operate for phase waves of a free particle, with  $p$  as its mementum along with the dirac  $\hbar$ , be expressed as;

$$k = \frac{p}{\hbar} = mv/\hbar$$

Then the wave function for a free particle, can be written as;

$$\Psi(r, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \varphi(k) e^{i(k \cdot r - \frac{\hbar k^2 t}{2m})} dk$$

Which implies that;

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t}.$$

However for a particle moving in a potential field  $\varphi(k)$  we have;

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + g\psi = i\hbar \frac{\partial \psi}{\partial t},$$

Where  $\varphi(k) \equiv g$  based on Newtonian equations of motion. Thus we get;

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + \varphi(r,t)\psi = i\hbar\frac{\partial(r,t)}{\partial t}$$

And this is the **Schrodinger wave equation**.

Then Born showed that  $\psi$  should be considered with its complex conjugate as  $\psi^*\psi dt$  and interpreted as the *probability* for a measurement of the position of the particle in volume element  $dr$  expressed as:

$$\phi(p) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{-i\mathbf{p}\cdot\mathbf{r}}{\hbar}} \psi(\mathbf{r},t) d\tau$$

This equation means that,  $\phi^*(p)\phi(p)dp_xdp_ydp_z$  is the probability that a measurement of the momentum of the particle, have  $x$ ,  $y$ , and  $z$  components of momentum falling within the intervals  $dp_x$ ,  $dp_y$ , and  $dp_z$ . Thus its average value can be expressed as triple integral as;

$$\bar{p} = \iiint \phi^*(p)\phi(p)dp_xdp_ydp_z$$

Now we consider something more interesting. The general solution of the wave equation is a superposition of plane waves as its individual components and frequencies. For instance for light and sound respectively. Thus we have transformations of waves which can be describe by Fourier transformations.

To go on we need to have a very brief discussion about this transformation mathematically then come back to physics.

We know that, a Fourier series is an infinite series rooted in trigonometric series and expressed as;

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Here  $a_n$  and  $b_n$  expressed as;

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \quad n \geq 1$$

Thus one can use Fourier series to "decompose a waveform into component waves of different frequencies and amplitudes". However it also able one to identify different sources of background or **random noise** such as in a signal. A noise is defined as random error or variation in observation which is out of model of a particular phenomenon.

Fourier transformation is an integral transformation. A relationship among two more functions can be expressed as;

$$f(x) = \int K(x, y) F(y) dy$$

"Then  $f(x)$  is the integral transform of  $F(x)$ , and  $K(x, y)$  is the **kernel** of the transform. If  $F(x)$  can be found from  $f(x)$  then the transform can be inverted" and we have inverse transformation. Integral transformations such as *Laplace* and *Fourier* transformations simplify differential equations and reduce them to linear equations with straightforward solution.

Thus **Fourier transformation** is an integral transformation and generally expressed as;

$$F(x) = \int_{-\infty}^{+\infty} f(x)e^{iyx} dx$$

The inverse Fourier transformation also expressed as;

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(y)e^{-iyx} dy$$

In a specific expression Fourier transform for wave can be defined as;

"a function  $\Psi$  of position and/or time can be resolved into exponential functions (waves) of strength  $\phi \exp(i\omega t)$ , where  $\phi$  depends on the wave vector and the frequency. this is expressed as;

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} \int \Phi(\omega)e^{i\omega t} d\omega$$

And

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int \Psi(t) e^{-i\omega t} dt.$$

Now we say Fourier transform of the equation;

$$\bar{p} = \iiint \phi^*(p) \phi(p) dp_x dp_y dp_z$$

Shows that it is equivalent to;

$$\bar{p} = \int \psi^*(\mathbf{r}, t) \left[ -\frac{\hbar}{i} \nabla \psi(\mathbf{r}, t) \right] d\tau,$$

Here  $\nabla$  is the gradient operator or Hamiltonian operator and the integral is over all 3-D space  $d\mathbf{r}$ .

This leads to the association of the operator

$$-(\hbar/i) \nabla$$

With the momentum  $p$  and to the following method for finding the average value of any dynamical variable associated with the particle. This variable is first expressed in terms of the momentum and position vectors of the particle and then the operator associated with the variable is obtained by replacing the momentum vector by the operator. If this operator is represented as  $\alpha$ , then the average value of the variable is;

$$\bar{\alpha} = \int \psi^* \alpha \psi d\tau.$$

It follows immediately that the operator;

$$H = -\left(\frac{\hbar^2}{2m}\right)\nabla^2 + \phi(r)$$

Corresponding to the Hamiltonian of the system, can be used to express the Schrodinger wave equation as;

$$H\psi = i\hbar(\partial\psi/\partial t).$$

It is also clear that operators constructed in this way must give real average values and are therefore Hermitian operators.

It is sometimes of interest to consider systems which are characterized by having a definite value  $E$  of the energy. This means that

$$H\psi = E\psi$$

The wave functions, i.e., the *energy Eigen functions*, or *energy eigenstates* of such states can be found by finding the Eigen functions  $U_E$  of the equation

$$HU_E = EU_E.$$

It is clear that  $U$  can be multiplied by any function  $f(t)$  and still satisfy this equation. Substitution of  $Uf(t)$  into the Schrodinger wave equation gives

$$f(t) = e^{-iEt/\hbar}$$

Thus the energy eigenstates are

$$\psi_E(\mathbf{r}, t) = U_E(\mathbf{r})e^{-iEt/\hbar}.$$

For a particle existing in such a state, the probability that a measurement of its position will give a value falling in the volume element  $dr$  is;

$$\psi_E^* \psi_E d\tau = U_E^*(r) U_E(r) d\tau.$$

Since this probability is independent of the time, such states are sometimes referred to as *stationary states*. In all work with wave mechanics, it is necessary to use wave functions  $\psi$  which are normalized i.e.

$$\int \psi^* \psi d\tau = 1$$

Since the only physical interpretation is in terms of probability, and for such an interpretation normalization is necessary. This means that the only solutions to the Schrodinger wave equation which have physical meaning are those for which;

$$\int \psi^* \psi d\tau, \quad \int \psi^* \frac{\partial^n \psi}{\partial x^n} d\tau, \quad \text{and} \quad \int \psi^* x^n \psi d\tau$$

Are finite, since the average values of all dynamical variables associated with the system must exist. When these conditions are satisfied there are systems for which the energy eigenvalue equation has acceptable solutions only for specific values of  $E$  over all or part of its range. In this case, the system is said to have *quantized energies* and these discrete energies in which the

system can exist are referred to as the *energy levels* of the system.

Here we end this part and refer the reader to the references and further readings.

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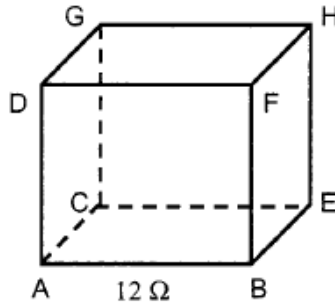
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## Physics riddles

Solving problems and challenges are human personalities. Some do this just for fun with crosswords, computer games and chess. Man frequently has to meet with the need of measuring phenomena and recording quantities in all domains of his life from science, technology, industry, agriculture to space travel or medicine. "Some of the major problems in physics are theoretical, meaning that existing theories seem incapable of explaining a certain observed phenomena or experimental result. The others are experimental meaning that there is difficulty in creating an experiment to test a proposed theory". Physics is mixture of diverse ingredients. No single method is adequate when dealing with Nature. Experiment and observation are vital. However this is true for concepts, pictures, imagination, mathematics, and physical intuition, and above all logical consistency. In other words, we need analytical thinking for solving physical problems since at the same time you must consider and combine several facts and find your way toward your answer. "We are explorers in a maze with mysteries at every turn—not for the faint of heart!" with this brief introduction we

present below some simple to harder problems in physics.

1. Twelve resistors, all with the resistance  $12\ \Omega$ , are connected to form the edges of a cube figure below.



What is the largest resistance between any two corners in the cube?

The largest resistance between two corners is  $10\ \Omega$ .

We first consider the resistance between the two most distant corners, such as, A and H in the figure. If a potential is applied between A and H, the corners B, C, and D will all be at the same potential, for symmetry reasons. Therefore we can connect B, C, and D and get three parallel resistors of  $12\ \Omega$  between A and (B, C, D).

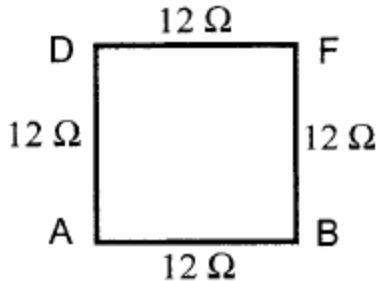
The resulting resistance is  $4\ \Omega$ . In the same way, the corners E, F, and G can be connected. The resistance between H and (E, F, G) is also  $4\ \Omega$ . Finally, we see that (B, C, D) is connected to (E, F, G) by six resistors of  $12\ \Omega$  in parallel, giving an effective resistance  $2\ \Omega$ . In total we get  $4 + 2 + 4 = 10\ \Omega$  between the corners A and H.

The problem asked for the largest resistance between *any* corners.

Thus we have to show that the resistance between two more close corners is less than  $10\ \Omega$ . Although the result is intuitively clear, it is instructive to go through the detailed arguments. If a potential is applied between A and E, the symmetry of the system implies that the corners (B,C) are at the same potential as the corners (F,G). Then there is no potential difference between G and C, or between F and B, and the links G-C and F-B can be taken out. We now have a network where the resulting resistance  $12\ \Omega$  for the link A-(B,C)-E is in parallel with the  $36\ \Omega$  of the link A-D-(G,F)-H-E. The total resistance between A and E thus is exactly  $9\ \Omega$ .

Finally, we consider the resistance between the two nearest neighbor corners A and B. This is a more complicated problem to solve exactly, but

as the original problem was posed, we only need to show that the resistance between A and B is *less* than  $10\ \Omega$ . Remove resistors so that only the network in figure below;



Then it remains, with  $12\ \Omega$  for A-B in parallel with  $36\ \Omega$  for A-D-F-B, which gives  $9\ \Omega$  between A and B. When resistors are removed the total resistance between any two points in a network must increase (or remain unchanged) because we are taking away some paths for the current. The resistance between A and B in the full cubic network cannot be larger than  $9\ \Omega$ .

**2.** Even in the absence of centrifugal effects from the Earth's rotation, the surface of the oceans would not have perfect spherical shape, because the mass distribution in the Earth does not have perfect spherical symmetry. Consider an underwater mountain, protruding from an otherwise almost flat seabed. Is the ocean surface

above that mountain slightly depressed or does it form a small hump?

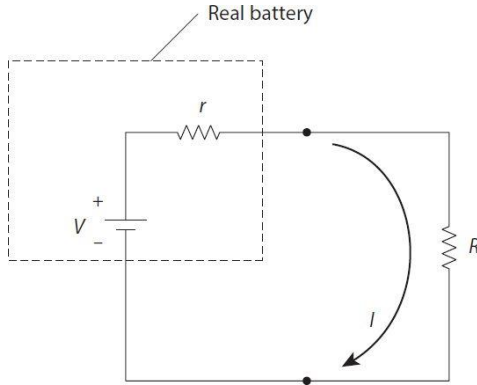
The ocean surface forms a small hump. It is a bumping surface in motion. The gravitational force between two bodies only depends on their masses and not on what they are made of or whether they are solid or liquid. If the underwater mountain is replaced by water, the ocean surface would be flat (neglecting the curvature of the Earth).



Because minerals have a higher density than water, the mountain pulls with an increased gravitational force on the ocean water. That force must be perpendicular to the ocean surface, since any force component parallel to the surface would set the water in motion. Thus the extra gravitational force from an underwater mountain gives rise to a small hump on the ocean surface.

**3.**A real battery (with internal resistance  $r > 0$  ohms), with a potential difference between its terminals of  $V$  volts (when no current is flowing

in the battery), is connected to a resistor of  $R$  ohms as shown in Figure;



What should  $R$  be so that maximum power is delivered to  $R$ ?

Solving by simple algebra we say;

The current  $I$  that flows by Ohm's law is;

$$I = \frac{V}{r + R}.$$

The power  $P$  dissipated (as heat) in  $R$  is (where  $E$  is the voltage drop across  $R$ )

$$P = EI = (IR)I = I^2 R,$$

Thus

$$P = V^2 \frac{R}{(r + R)^2}.$$

Clearly,  $P = 0$  when  $R = 0$ , and  $P = 0$  when  $R = \infty$ . Thus, there is some  $R$  between zero and infinity for which  $P$  reaches its greatest value. This value can easily found by differentiating  $p$  with respect to  $R$  and set the result to zero. However as we said before we use simple algebra here instead.

$$P = V^2 \frac{R}{r^2 + 2Rr + R^2} = V^2 \frac{R}{r^2 - 2Rr + R^2 + 4Rr}$$

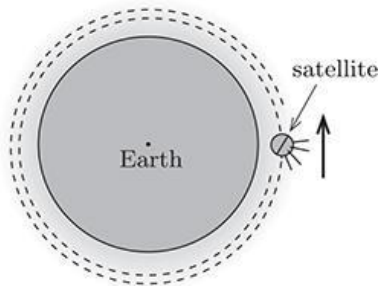
$$= V^2 \frac{R}{(r - R)^2 + 4Rr} = V^2 \frac{1}{\frac{(r-R)^2}{R} + 4r}.$$

We maximize  $P$  by minimizing the denominator of the RHS of this equation, which occurs when  $R = r$ . this is done because that makes the first term in the never negative denominator as small as possible, i.e equal to zero. Thus,  $R = r$ , and the maximum power in  $R$  is;

$$\frac{V^2}{4R}$$

**4.** A half-ton satellite is orbiting the Earth on a roughly circular orbit when it is abandoned. The drag acting on this particular satellite can be expressed as  $cpv^2$ , where  $c = 0.23 \text{ m}^2$ ,  $\rho$  is the

density of air at the altitude of the satellite and  $v$  is the speed of the satellite.



Note: Because of the effects of air drag, abandoned satellites, at the end of their useful lives, lose energy in the upper layers of the atmosphere, before finally burning up when they reach the denser lower layers. It can be shown that satellites originally moving along circular trajectories will continue to travel in approximately circular orbits, with their orbital radii slowly decreasing.

**a)** Does the satellite brake or accelerate as a result of the air drag? How can your answer be explained from the point of view of dynamics? **b)** A simple connection can be found between the drag force and the tangential acceleration of the satellite. What is it? **c)** What is the density of air at an altitude of 200 km, if in this region the orbital radius of the satellite decreases by 100 m during a single revolution?

**a)** The trajectory of the satellite passing through the (thin) atmosphere can be assumed to be

circular throughout, and so the dynamical condition governing the motion is;

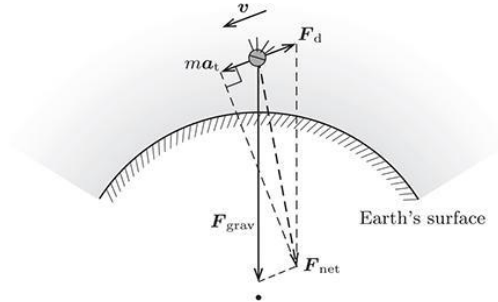
$$G \frac{mM}{r^2} = m \frac{v^2}{r},$$

Where  $m$  and  $M$  are the masses of the satellite and Earth, respectively,  $r$  is the distance of the satellite from the center of the Earth, and  $v$  is the speed of the satellite. From this we get the expression for  $v$  as

$$v = \sqrt{\frac{GM}{r}},$$

Which shows that, if the altitude (the orbital radius) of the satellite decreases, then its speed increases, i.e. the satellite *speeds up* as the result of any air drag. This surprising fact is usually known as the *astronautical paradox*. The speed increase of the satellite may also be understood dynamically with the help of the (not-to-scale) figure. The drag force  $F_d$  is directly opposed to the velocity, and the gravitational force is directed towards the center of the Earth. However, because of the (slow) decrease in the satellite's height, the latter is *not* perpendicular to the satellite's velocity. The non-zero tangential component of the gravitational force is in the same direction as the velocity. If (as will be

shown in part *b*) to be the case) this component is larger than the drag force, then the satellite will speed up, rather than slow down.



*b*) The (formal) power of the drag force is negative,  $F_d \cdot v = -F_d v$ , and this is to be equated with the rate of total energy loss of the satellite:

$$\frac{dE_{\text{total}}}{dt} = -F_d v.$$

The total energy is the sum of the kinetic energy and the gravitational potential energy:

$$E_{\text{total}} = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}mv^2 + \left(-G\frac{mM}{r}\right).$$

Using the equation;

$$G\frac{mM}{r^2} = m\frac{v^2}{r},$$

The connection between the kinetic and potential energy is always the following:

$$E_{\text{pot}} = -G \frac{mM}{r} = -mv^2 = -2E_{\text{kin}},$$

And so we can write  $E_{\text{total}} = -E_{\text{kin}}$ . It follows that any *decrease* in the total energy of the satellite results in the same *increase* in its kinetic energy.

With the help of this equation, statement;

$$\frac{dE_{\text{total}}}{dt} = -F_d v.$$

Can be transformed to;

$$-\frac{dE_{\text{kin}}}{dt} = -F_d v,$$

And using the usual expression for the kinetic energy:

$$F_d v = \frac{d\left(\frac{1}{2}mv^2\right)}{dt} = mv \frac{dv}{dt}.$$

The expression  $dv/dt$  on the RHS is just the tangential acceleration  $a_t$  of the satellite (along the trajectory), so

$$F_d = ma_t \text{ or using vector notation } -\mathbf{F}_d = m\mathbf{a}_t.$$

In accordance with Newton's second law, the expression  $m\mathbf{a}_t$  must be equal to the tangential component of the net force  $\mathbf{F}_{\text{net}}$  exerted on the satellite. Here, this is the sum of the drag force

and the tangential component of the gravitational force (*see* figure). It then follows from equation;

$$F_d = ma_t \text{ or using vector notation } -F_d = m\mathbf{a}_t.$$

That the tangential component of the gravitational force (increasing the satellite's speed) is exactly twice as large as the drag force (causing energy loss). So, the speed of the satellite is increased by a force that is equal to the drag force in strength, but *oppositely* directed.

The perpendicular component of the gravitational force causes the change in direction of the satellite's velocity and the curvature in its trajectory, but the magnitude of the velocity (the speed) is not changed by this.

c) Consider again the change in the total energy of the satellite, but now expressing it in terms not of kinetic energy but of gravitational potential energy. Using equation;

$$E_{\text{pot}} = -G \frac{mM}{r} = -mv^2 = -2E_{\text{kin}},$$

And the deduction from it, the connection between the small changes in the total and potential energies can be written as

$$\Delta E_{\text{total}} = \frac{1}{2} \Delta E_{\text{pot}}.$$

The change of the potential energy can be expressed in terms of the small change of the orbital radius  $\Delta r$  as follows:

$$\Delta E_{\text{pot}} = \Delta \left( -G \frac{mM}{r} \right) = G \frac{mM}{r^2} \Delta r.$$

We know that, during a single revolution (lasting  $2\pi r/v$ ), the distance of the satellite from the center of the Earth is decreased by  $\varepsilon = 100$  m, so the rate of change of the orbital radius of the satellite is;

$$\frac{\Delta r}{\Delta t} = -\frac{\varepsilon}{2\pi r/v}.$$

Thus the drag force acting on the satellite can be expressed as;

$$F_d = \frac{1}{4\pi} G \frac{mM}{r^3} \varepsilon.$$

Now using equation;

$$G \frac{mM}{r^2} = m \frac{v^2}{r},$$

And the fact that the drag force has the form  $F_d = c\rho v^2$ , the atmospheric density at the altitude of the satellite is found to be;

$$\rho = \frac{1}{4\pi c} \frac{m}{r^2} \varepsilon.$$

Inserting the given data for  $\varepsilon$ ,  $c$  and  $m$ , and the value of the orbital radius  $r = 6370 \text{ km} + 200 \text{ km} = 6570 \text{ km}$  gives;

$$\rho = 4.0 \times 10^{-10} \text{ kg m}^{-3}.$$

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## **BOOK THREE**

### **Computer**

In this book we tend to show that how the subject of mathematics, physics and computing works with each other. Therefore the previous structure will be mixed here and we have a book with one part not four but containing four. Another reason is that the most important basic blocks of computer and computing science lies in mathematics, logic and physics in software and hardware operations and we discussed them in two previous books.

We know that everything in this universe; natural or man-made or UFO-made from atom to galaxies are doing a function since they accept or insert inputs then doing something on them and give outputs. Thus the universe is a machine composed of sub-machines; a function composed of sub-functions a set composed of subsets an existence composed of sub-existence as Persian scholar Moala Sadra says.

Calculating machines are thinking machines because they behave like a brain. They can carry out numerical calculations, geometrical sketching, image and data processing, manage traffic and probe space crafts.

Thus they do things from simple counting to solving differential equations of our models for different phenomena of nature and daily life, commerce and market. **In fact they just receive and follow instructions.** This is important to know since they are not responsible for any error! They store information in their memories then carry out the logical operations we call deductive reasoning as we mentioned in the book one in mathematical thinking. In fact almost everything about computational thinking is the same with mathematical thinking.

Therefore the machines have brain properties. "They can make comparisons, and then, on the basis of what they observe, choose among alternative courses of action". A playing machine, can predict several moves in advance. Then, using rules of probability, logic and math as different parts of its instructions, it evaluates the different possible moves, and chooses the best one.

Machines can learn from their experience as well. In the case of play games firstly they are guided by its original instructions. Then they start to store a record of all their moves in their memories. After a certain number of games they evaluates their moves in terms of their actual consequences gained from their past experiences

by modifying first instructions to new most efficient ones.

It is natural to say to the brain is like a calculating machine. This is right since we are doing such logical calculations from the very first beginning days of our life until the end.

A digital calculating machine is a network of complex electrical units. The basic unit is the **vacuum tube, the transistor**. There are also other units such as CPU's, and gates, RAM's, and inverters. In comparison the brain is also a network of complex electrical units. The basic unit in the brain is the **nerve cell**. The behavior of many nerve cells is something like that of a CPU in stimulating electrical pulses.

The word **algorithm** in its Latin form appeared for the first time in 1202, in a treatise called Liber Abaci by Leonardo Fibonacci. Today, its equivalent is coding.

Algorithm and coding indicate how to construct a series of actions and instructions to achieve a desired result or solve a problem.

Algorithms and coding generally constitute of delicate arrangements (sequences) of commands written for a machine, such as computers, devices, vehicles, doors, elevators etc.

An **algorithm** is the strategy for "designing and evaluating" a series of single actions, while

"**coding** reflects the operational phase that leads to the execution of those actions on a particular computing device", such as a PC and smartphone. A **search algorithm** is very similar to our alphabetical searching for a word in a dictionary text book.

Using machines we accessed today we must precisely specify every operation. Among these machines computers experienced remarkable successive development of levels in algorithms and coding. In fact this precision must be applied on their explicit "internal structure" from written in **machine language** for technicians, to those in **high-level computer languages** known as **programming**.

Comparison between the brain and computer is not a trivial one. In fact they are coworkers in their scientific studies. Thus this comparison have already shown themselves to be useful. **John von Neumann** and his co-workers were designing the computer known as the JONIAc, imitating brain characteristics. While, those who were examining the brain function, i.e. neurologists, follow the lines of computer designers and investigators. Thus there exist a parallel investigation trend between them such that one can say, "the more we develop computers, the better we shall understand them. And the better we understand our

computers, the better we shall understand ourselves".

Computer science have composed of *programming, hardware, social issues* of computing, and computer *skills*. However **computational thinking** is different and more than these basic computing knowledge. It consider computer science "as an independent body of thought". Algorithmic thinking is similar to it and also need, investigation, creativity, proof etc. such as *mathematical thinking*.

However software engineers use diagram technics for their analysis to get their best strategy. A computer machine works with digitized data not raw data in any subject from music and art to space exploration and science. It represent the final result for analysis by graphics using different modeling technics. Consequently in today research area, understanding "software requirements and limits of computing" is very essential.

Computational thinking is the way that computer scientists think and reason. However as we mentioned above our scientific community today, strongly depend computing knowledge of the researchers. Thus even a non-computer scientist or researcher must know how to think computationally. "As a result, new specialties,

such as computational biology and computational physics, have become common in institutions of higher education".

To start an elemental discussion or story about a computer the best point is a simple ancient computational tool called abacus. In comparison with modern computer, an abacus use four concept of it:

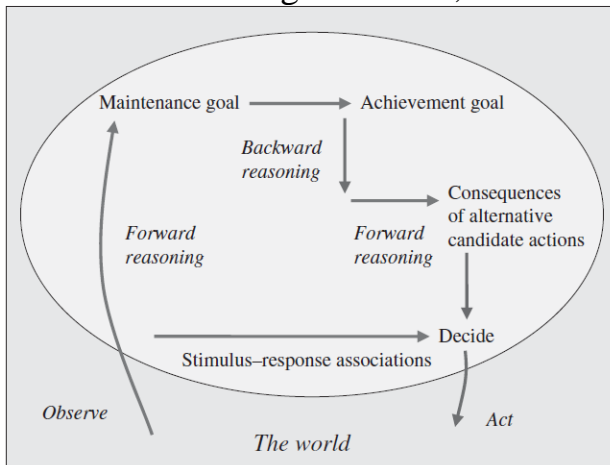
1. Storage
2. Representation
3. Calculation
4. User interface

If there is something to store those items referred to as *data*. "An abacus can only store a single datum at any point in time, while your laptop can store trillions of pieces of data". A representation means to model something else based on data provided as information. The abacus represents an integer while a modern computer is able to solve real-world problems based on recorded data as digital numbers (bites) and in near future as quantum numbers (qubits) provided as information.

"Neither the abacus nor computer hardware alone can perform calculations". Abacus's beads must be pushed around by a human while computer hardware needs software's, to perform calculations.

**User interface** is human's communication traits with the machine which is happen both in an abacus and a computer.

All programs, have "a relatively small number of distinct fundamental building blocks". Using logical thinking you make logical expressions in codes to have logical decisions in programing. You "even chain them together using logical operators". Logical decisions are combined of basic algorithmic constructs such as **if-statements** and **loops**. Algorithms able you "to track of data that are important to your solution that is, the **state**". Mismanagement of stare leads to bugs. The different data types the different data structure. The Computational Logic can be pictured as this in an agent's mind;



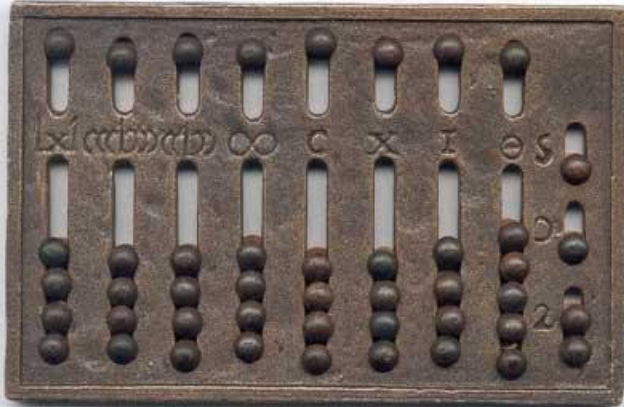
The agent's mind is a **syntactic structure**, which represents the agent's belief and goals about the world as it is and would be. His/her sentences about them have a **conditional syntactic form**. "The world, is a **semantic structure**, which includes the agent's body, and gives meaning to the agent's thoughts". In fact, universe is a *dynamic structure*, which is continuously changing, and exists only in the here and now. However, "the agent can record its changing experiences in its language of thought, and formulate general beliefs about the causal relationships between its experiences. It can then use these beliefs, which explain its past experiences, to help it achieve its goals in the future".

Thus usually an agent observes events and their properties in the world and using forward reasoning based on those observations as well as its intuition and past experiences, derive conclusions expressed in the logical form of conditionals. Then the agent must decide among alternative plans.

Consequently, one can say, benefits of computational Logic, which have had some success in the field of Artificial Intelligence, also have great potential for improving human thinking and behavior.

Human being used sticks, stones, and bones as counting tools as we discussed in the first book. Then in the course of time computing devices were developed over time. As we mentioned in the previous part a computer is an electronic machine that accumulates, stores and processes information, based to user instructions, and then yields the outcome. Thus it is a "programmable electronic device that performs arithmetic and logical operations automatically using a set of instructions provided by the user" called *program*, *code* or *algorithms*. These three terms can be used with slight difference.

**Abacus** was invented by the Chinese around 4000 years ago. They called it *Suan-Pan* in Chinese.



It's a rectangular wooden frame with metal rods with beads attached to them. The abacus operator

moves the beads according to certain guidelines to complete arithmetic computations.

As we mentioned above computing devices have been one of the greatest technological achievements along history. However the most remarkable development accrued in twentieth century, which leads to advent of personal computer, in 1970s. In fact on that time human being have reached a new stage in the evolution of the great idea that began with the abacus. Thus one can say computer device is, "the reflection of our minds".

At first point the history of computers started with the invention of the abacus while in the second point back to the design of ENIAC. However more fair people back the computer idea to the Elamites and Babylonians and their clay tablets of algebraic algorithms and astronomical data 5000 years ago. Thus we traveled from those clay tablets and sliding beads of the Chinese abacus to a machine that could preserve and perform a series of instructions called *program*.

The creation of logarithms, or logs as the inverse operation of power by Scottish **John Napier** (1550-1617), in 1614 was one of the important triumphs in the development of mathematics and computers. He also prepared some tables of log value for natural numbers. However in 1617,

**Henry Briggs** (1561-1630), published some tables "giving the logs for the numbers from 1 to 1,000 and, seven years later, a much larger one for 2,000 to 29,000 and 90,000 to 100,000". His tables were circulated widely on that time since they made the arithmetic easier if not much. Gradually the gaps filled in his tables, by others mathematicians. They found logs for handy mathematical functions, such as sine and tangent. This made Napier's invention an increasingly indispensable tool for **navigators** and surveyors. This invention has a great deal of influence on the development of mathematics and computers as well.

His look on numerical *powers*, led to another development i.e. development of *exponents*.

A device called **Napier's bones**, was invented by Napier himself at the end years of his life. It was a handy calculating apparatus do arithmetic more than abacus. "It uses 9 separate ivory strips called bones marked with numerals to multiply and divide. It was also the first machine to calculate using the decimal point system". Below we have the way of multiplication operations with the device.

Diagram illustrating the multiplication of two 9-digit numbers using a 9x9 grid. The grid contains digits 1-9. Lines connect specific digits in the grid to the corresponding digits in the multiplication problem  $46785399 \times 96431$ .

The multiplication problem is shown as:

$$\begin{array}{r} 46785399 \\ \times 96431 \\ \hline \end{array}$$

The intermediate steps and the final result are shown below the grid:

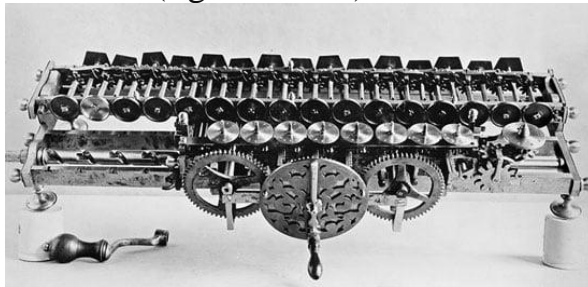
$$\begin{array}{r} \rightarrow 46785399 \\ \rightarrow 140356197 \\ \rightarrow 187141596 \\ \rightarrow 280712394 \\ \rightarrow + 421068591 \\ \hline 4511562810969 \end{array}$$

In 1620, the English mathematician **William Gunter** (1581-1626) made another development on the way. He "drew a grid of lines on a sheet of parchment and multiplied and divided numbers by adding and subtracting lengths with a compass". The operation principle was still based on logs and exponent, in the way that, "each point on Gunter's scale, or line, is exponentially distant from the others". The tool was called "**the Gunter**" which was a popular navigator's tool.

In 1622 another English mathematician, **William Oughtred** (1574-1660), "rearranged Gunter's lines into a pair of circles, refigured their numbers", and designed a useful device for scientists and engineers. In this rout of progress for navigating and computational tools, **Pascaline** was invented in 1642 by **Biaise Pascal**, a French mathematician and philosopher. It is thought to be the first mechanical and automated calculator. It was a wooden box with gears and wheels inside as is showed below.



In 1673, a German mathematician and philosopher named **Gottfried Wilhelm Leibniz** improved on Pascal's invention to create this apparatus. It was a digital mechanical calculator known as the **stepped reckoner** because it used fluted drums instead of gears. It also called Leibniz wheel (figure below).



A steam-powered mechanical computer that could do basic computations such as logarithmic tables invented by **Charles Babbage** in the early 1820s. It is called **Difference Engine**.

He created another calculating machine, called **Analytical Engine**, in 1830. The punch cards provide input for this device. One can call it the first computer since "it was capable of solving any mathematical problem and storing data in an indefinite memory". It contained an ALU, some basic flow chart principles and the concept of integrated memory".

In the year 1890 an American Statistician **Herman Hollerith** invented a Tabulating Machine. It was similar to Babbage's Analytical Engine and a mechanical tabulator. "It could compute statistics and record or sort data or information. Hollerith began manufacturing these machines in his company, which ultimately became International Business Machines **IBM** in 1924".

In 1930 **Vannevar Bush** presented the first electrical computer, the **Differential Analyzer**. It was made up of vacuum tubes that switch electrical impulses in order to do calculations similar to our brain. "It was capable of performing 25 calculations in a matter of minutes".

Now it was the time to go with enormous numbers and massive calculation. In 1937 **Howard Aiken** intended to invent a machine that could do this task. "**The Mark I** computer was constructed in

1944 as a collaboration between IBM and Harvard".

In fact most of the earliest computers developed in the United States, Britain and Germany. The Second World War and the need to cryptography for military usage led to digital computers to do faster methods of computation. They were "large bulky machines consisting of several thousand vacuum tubes". Thus they occupy the space of a large room. Even with their development they were "slow and unreliable" yet. However Machines such as "**Harvard Mark I** were large electromechanical calculators that could perform mathematical calculations quickly".

A machine called **ABC** designed to solve a set of linear equations was presented by Atanasoff and Berry. While "Mauchly and Eckert designed the **ENIAC** and **EDVAC**, and the **ENIAC** needed to be physically rewired to solve different problems".

**EDVAC**, was able to store its instructions i.e. program, to solve a different problems. This meant that "a new program was loaded into the memory of the machine" then it solve the new problem.

The **Colossus computer** developed in a "codebreaking" project by a team at Bletchley Park in England. "This allowed them to crack the

German Lorenz codes and to provide important military information during the D-Day landings of 1944".

In Germany, Konrad Zuse developed **Z3** machines in 1941. It was the world's first programmable computer. While Williams, Kilburn and others implemented the first stored-program computer called **Manchester Baby**. This technology developed and we have smaller and faster computers. The first **laptop** in 1981 introduced by Adam Osborne and EPSON.

We considered above a very general and quick history of computing. Now we go on with the five generation of modern computers. Not is that, we also have five generation of computer languages which will come along later.

The *first generation* belongs to a period of 1940 to 1955. In this period machine language was developed. Vacuum tubes for the circuitry, magnetic drums for the memory, punch cards and magnetic tape and paper tape for output and input have used in this period. They includes machines such as ENIAC and EDVAC.

The *second generation* of computers started to be developed in 1957-1963. Computer languages such as COBOL and FORTRAN are employed in this period. Vacuum tubes advanced transformed to **transistors**. "This made the computers smaller,

faster and more energy-efficient". Binary languages became assembly languages. They include IBM 1620, IBM 7094, etc.

The *third generation* of computers belongs to 1964-1971. An innovation made up of many transistors called Integrated circuit (IC) developed in this period. This ICs could increase the power of a computer with the lower cost. "High-level programming languages such as FORTRAN-II to IV, COBOL, and PASCAL PL/1" were developed and utilized in this time. They include machines such as; IBM-360 series, and the Honeywell-6000 series.

The *fourth generation* (1971-1980) of computers started with the invention of **microprocessors**. In this time **personal computers** began to be produced. Program languages such as C, C++ and Java were developed and used. Machines for this period are such as; STAR 1000, PDP 11, CRAY-1, CRAY-X-MP, and Apple II.

Since 1980 the *fifth generation* of computers were developed and used until present time which we are going to start the generation of quantum computers. This is the period of Artificial intelligence AI, parallel processing using **superconductors** instead of previous **semiconductors** and using "Ultra Large Scale Integration (ULSI) technology". C, C++, Java,

Python, Net...are among the most useful programming languages. Machines such as IBM, Pentium, Desktop, and Laptop belongs to this period.

The tireless inventors of computing machines encountered with lots of ongoing problems in design and implementing of their machines to get them into work especially from 19<sup>th</sup> century.

According to above we must have different types of computers. We consider them briefly below.

**"Analog computers** are built with various components such as gears and levers, with no electrical components. One advantage of analogue computation is that designing and building an analogue computer to tackle a specific problem can be quite straightforward.

Information **in digital computers** is represented in discrete form, typically as sequences of 0s and 1s (binary digits, or bits). A digital computer is a system or gadget that can process any type of information in a matter of seconds. Digital computers are categorized into many different types. They are as follows:

It is a computer that is generally utilized by large enterprises for mission-critical activities such as massive data processing. **Mainframe computers** were distinguished by massive storage capacities,

quick components, and powerful computational capabilities. Because they were complicated systems, they were managed by a team of systems programmers who had sole access to the computer. These machines are now referred to as servers rather than mainframes.

The most powerful computers to date are commonly referred to as **supercomputers**. Supercomputers are enormous systems that are purpose-built to solve complicated scientific and industrial problems. Quantum mechanics, weather forecasting, oil and gas exploration, molecular modelling, physical simulations, aerodynamics, nuclear fusion research, and crypto analysis are all done on supercomputers.

A **minicomputer** is a type of computer that has many of the same features and capabilities as a larger computer but is smaller in size. Minicomputers, which were relatively small and affordable, were often employed in a single department of an organization and were often dedicated to a specific task or shared by a small group.

A **microcomputer** is a small computer that is based on a microprocessor integrated circuit, often known as a chip. A microcomputer is a system that incorporates at a minimum a microprocessor, program memory, data memory,

and input-output system (I/O). A microcomputer is now commonly referred to as a personal computer (PC).

These are **miniature computers** that control electrical and mechanical processes with basic microprocessors. Embedded processors are often simple in design, have limited processing capability and I/O capabilities, and need little power. Ordinary microprocessors and microcontrollers are the two primary types of embedded processors. Embedded processors are employed in systems that do not require the computing capability of traditional devices such as desktop computers, laptop computers, or workstations".

As we said before we have three kinds of computer; *analogue*, *digital* and the new improving one i.e. *quantum* computers. But as this book is about mathematics, physics and computing and we said lots of the related elements to this part in our previous books thus we do not consider them again here. Therefore our elements of computing is a mixture of related mathematical and physical basis. However we do not go deeply and refer the reader to the further reading and references.

It is hard to think of modern society today without modern technology such as smart phones, laptops,

Skype, Twitter and WhatsApp. While in preceding generations, communication was existed by writing letters that often took months to reach the recipient. As we mentioned before a computer is a programmable and data storable electronic device. It involves two basic parts, of hardware and software. "The hardware is the **physical** part of the machine, and the components of a digital computer include memory for short-term storage of data or instructions; an arithmetic/logic unit for carrying out arithmetic and logical operations; a control unit responsible for the execution of computer instructions in memory; and peripherals that handle the input and output operations". While the Software part "is a set of instructions that tells the computer what to do" in a logical and mathematical way.

The earliest analog computers run on quite different principles than of digital ones. The representation of data in an analog computer reflects the properties of the data that are being modelled. For example, data and numbers may be represented by physical quantities such as electric voltage, whereas a stream of binary digits is used to represent them in a digital computer.

A digital computer operates based on binary format i.e. binary digits of zero and one. It is similar to being on or off in a switch. In fact "a

digital computer is a sequential device that generally operates on data one step at a time". In earlier types, "a single transistor (initially bulky vacuum tubes) is used to represent or store a binary digit". The Larger numbers the more transistors. Thus "a stream of binary digits" while in an analog computer "data and numbers may be represented by physical quantities such as electric voltage".

One of the most fundamental architecture was of the Von Neumann's. It was applied for early digital computers. It needed a single store for machine instructions and data to reconfigure and perform different task. With this architecture a user could "enter new machine instructions in computer memory rather than physically rewiring a machine as was required with ENIAC".

Nature is analog rather than being digital while it is mathematical language that can describe its relationships from quantum to galaxy. Based on this we can build mathematical model of universe. "This is the key insight that leads to the analog computer". The simplest analog computers use physical components that model **geometric ratios**. The earliest known analog computing device is the **Antikythera** Mechanism. Constructed by an unknown scientist on the island of Rhodes around 87 B.C this device used a

precisely crafted differential gear mechanism to mechanically calculate the interval between new moons (the synodic month). Interestingly, the differential gear would not be rediscovered until 1877.

**James Thompson** brought foundational of analog computation in the nineteenth century using wheel and disc integrator, to perform the integration of a product of two functions.

He even an analog computer for tide predicting. By parallel operations an analog computer could simulate dynamical system. It was **Vannevar Bush** at the MIT who developed the first general large-scale analog computer. This machine was **Bush's differential analyzer**. In fact it was designed "to solve 6th-order differential equations by integration".

For doing this task it used considerable amount of wheels, discs, shafts, gears, motors and miles of wires connecting relays and vacuum tubes in addition to a considerable set-up time. "Data representation in an analog computer is compact, but it may be subject to corruption with noise. A single capacitor can represent one continuous variable in an analog computer, whereas several transistors are required in a digital computer".

As we said before, early digital computers used vacuum tubes to store binary information, using binary value of '0' or '1'.

The fundamental architecture of a computer has remained basically the same since von Neumann and others proposed it in the mid-1940s. It includes a central processing unit which includes the control unit and the arithmetic unit, an input and output unit and memory.

Smaller, cheaper and reliable than the vacuum tubes were **transistors**. They are fundamental in modern electronic. They are three-terminal, solid-state electronic devices. They use "to control electric current or voltage between two of the terminals by applying an electric current or voltage to the third terminal". Using these transistors, one can have an electric switch. Having a series of these switches, controlling each other one can build a "complicated logic circuits". Then can go on to have a very compact "silicon chip with a density of a million transistors per square centimeter". Here these switches may be turned on and off very rapidly e.g. every 0.000000001 per second. And as we know chips operate as a brain for modern electron devices.

As we said before transistors were developed after the Second World War. Their invention happened in research for a solid-state alternative to bulky

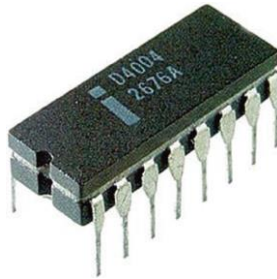
unreliable vacuum tubes. Three inventors of transistors i.e. Shockley, Bardeen and Brattain were awarded the Nobel Prize in Physics in 1956. However it was **Jack Kilby** invented the **integrated circuit** in 1958. The integrated circuit was a solution to the problem of building a circuit with a large number of components. He won and the Nobel Prize in Physics in 2000 for his contribution.

It was based on the idea of having several transistors at the same time on the same piece of semiconductor instead of separate ones. "This allowed transistors and other electronic components such as resistors, capacitors and diodes to be made by the same process with the same materials".

An *integrated circuit* is a set of electronic circuits on a small chip of semiconductor material. Today they contain billions of transistors in a tiny area. The width of each conducting line now is measured in tens of nanometers. In fact they are based on miniaturizing transistors and placing them on silicon chips called semiconductors. This process is known as Moore's law after Gordon Moore. He was one of the founders of Intel who formulated the law in the mid-1960s. Use of semiconductors give birth the third generation of computers i.e. devices along with the mouse,

keyboard -I'm writing on it- and monitors working with an operating system. These devices are able "to run many different applications at one time with a central program that monitored the memory".

In 1969 the world's first **microprocessor** at chip level was Intel 4004.



It was a general-purpose logic device that derived its application instructions from the semiconductor memory. The development of these devices give birth to the fourth generation of computers. They contain thousands of integrated circuits which placed into a single silicon chip. Today a single chip could contain all of the components of a computer from the CPU and memory to input and output controls. The fourth generation of computers could fit in your palm while the first generation of computers filled an entire room.

As we mentioned earlier a computer program is a set of statements. But this statements and instructions in every form i.e. analog, digital or quantum, must have a logical and mathematical basis which is called **foundations of computing**. From starting point this include basically working with binary number system and Leibniz's step reckoner calculating machine and Babbage's difference engine. In fact, Babbage's design of the analytic engine provided the vision of a modern computer however working with a designed pattern of cards. With this machine we also have the world's first programmer i.e.

Lady Augusta Ada Lovelace. Her program was a plan of calculating Bernoulli numbers by the engine. She was a mathematician collaborating with Babbage. We have a program language called **Ada** named in her honor. It was Boole's symbolic logic that provided the foundation for digital computing. In fact he worked on "formulating a calculus of reasoning". He showed that logic is bridge between mathematics and philosophy by converting Aristotle's syllogistic logic to a set of algebraic equations. As we mentioned before *Boolean algebra* is based on number 1 and number 0. In fact in Bool's mind; number 1 represent, "the universe of thinkable objects" we called it the *universal set* if you

remember. The number 0 represents "the absence of any objects" we called it the *empty set*. After this introduction, he used algebraic symbols such as  $x, y, z$ . He did this to represent "collections or **classes** of objects" instead of adjectives and nouns i.e. **meanings** in human language. After this step, well, he performed, algebraic operation on those classes of objects to combine them similar to human language. Thus every meaning in human language converted to a class of objects in Boolean logic. His idea works based on operations similar to the set of real number system based on set theory which we explained in the book one.

Therefore Boole's idea which published in his book as *The Mathematical Analysis of Logic* 1847 and *An Investigation of the Laws of Thought* 1854 showed that we can build a machine behave similar to our mind. We know that our brain consists of some physical compartments transmitting electrical pulses and signals containing information, received in present time by our inlets i.e. ears and eyes or saved information from past. we also able to do some predictions from our experiences and this is called **artificial intelligence** or **machine learning** which is an active field of research in mathematics and technology today. So we tried to build something to imitate our brain founded on

transformation of logic into mathematics. As we mentioned earlier, switches represent binary values is the foundation of modern computing. It works as below;

"A high voltage represents the binary value 1 with low voltage representing the binary value 0. A silicon chip may contain billions of tiny electronic switches arranged into logical gates. The basic logic gates are AND, OR and NOT. These gates may be combined in various ways to allow the computer to perform more complex tasks such as binary arithmetic. Each gate has binary value inputs and outputs".

But *how* and *who* changed this conversion of logic to thinking machine?

An American mathematician and engineer **Claude Shannon** showed the application of Bool's algebra provides a "perfect mathematical model for switching theory" applying in devices such as telephones and also for the "subsequent design of digital circuits and computers". He along with his supervisor Vannevar Bush at MIT give birth the digital computing based on Boolean algebra. Bush proposed that, how the binary digits of 0 and 1 can be represented by electrical switches. He implied number 1 for true or YES and number 2 for false or NO and this abled relay circuits to perform binary arithmetic and more

complex mathematical operations. In fact this invention "provided electronic engineers with the mathematical tool they needed to design digital electronic circuits and provided the foundation of digital electronic design and telecommunication systems". This is the process going on even now in digital computers. We said this due to the space and we refer the reader to the references and farther readings.

As we seen in the development process of digital computers, mathematicians, physicists, designers and engineers tried to get to a faster and more efficient computing device. Now we say. Some people said. If Aristotelian point of view to the world i.e. his philosophy inverted to a logical mathematical language by Bool. And that applied in designing of electrical switches using binary system of numbers. This give us these computer devices we have today. Then it is natural for one to say we can use the new developed quantum language of the world to this subject and have a revolution in computing devices. These devices are called quantum computers.

**Quantum computer** is a computer based on a computational model using *quantum mechanics*. "In fact Quantum computing is the study of ways in which unusual quantum mechanical effects

could be employed to improve the processing power of computers in solving problems". This idea of quantum computers have been presented by Feynman in 1981. He received the Nobel Prize in Physics in 1965, for his fundamental work in quantum electrodynamics which deeply affected the research on *elementary particles*.

He asked, "Can physics be simulated by a universal computer, i.e. **Turing Machine** or Neumann-type computer?"

He proposed that, since the fundamental laws of nature are *reversible* laws of thermodynamic then classical computers, were not compatible with them since governing laws of those computers were *irreversible*. Thus "we need a new computer, which is different from Turing Machine and obeys the principles of quantum mechanics, i.e., quantum computer".

This is also on the fact that, reversible computations satisfy the laws of thermodynamics. There exist two kinds of reversible process i.e. physical and logical. In the first, entropy does not increase while in the second the process can construct the inputs from the outputs. Thus there exist a cycle of matter and information. Consequently, the physical reversible process the logical reversible process.

Along with the development in computer technology it will be possible to design computer systems with components in size of an atom or atoms. Maybe we can have a "storage system in which two bits of information were stored using two different quantized spin states of an atomic nucleus".

We also can have a "logic gate formed of a small number of atoms on the surface of a substrate". This is called, **Nano computing**. However this devices use quantum effects but they are not referred as quantum computers.

This is due to the fact that, the so-called quantum computers and computing, involves more characteristic of quantum systems. This statement is because, one can say, the quantum computing process is started in 1985 by David Deutsch with his "theoretical paper on the idea of a universal quantum computer".

In a quantum mechanical system, such as an atom or *elementary particle* we may have two or more distinct states. These electron might have two different spin states. A photon might have two different polarizations.

Formally, a particle, can exist in an indeterminate **superposition** of the two states simultaneously. But if the particle choose one of the states, then

the related wave equation will collapse. In fact there will be a reduction in state vector.

In a thought experiment called EPR experiment, Einstein, Podolsky, and Rosen in 1935 showed a quantum phenomenon called quantum **entanglement**. It express that;

"Suppose two different particles are created from a single particle and move in different directions. It is known that they must have opposite spins, because spin is conserved, but the spins of both are taken to be indeterminate until a measurement is made. If the spin of one particle is measured, then its spin is fixed. But the measurement on one particle instantly fixes the spin of the other, even though the particles may be widely separated. The two particles are effectively part of the same system and are said to be in an 'entangled state'".

Both of these quantum effects i.e. quantum superposition and entanglement are using in Quantum computing. According to superposition effect, "a system can exist in two superposed states constituting a **qubit**, i.e.  $(0, 1)$ ". However, in the case of collapse of the related wave equation, this qubit also "collapses to one of two real states 0 or 1 corresponding to classical bits". Therefore a "superposed state can store the bits 0 and 1" simultaneously.

A classical computer **register** bits can store each of eight binary system numbers at a time. These numbs include, "000, 001, 010, 011, 100, 101, 110, and 111". While "a register made of three qubits actually stores all eight numbers simultaneously".

Consequently any quantum computation process on all possible numbers in the register can occurs simultaneously. The more qubits the more registers. This is an exponential trend. Thus if we imply Four qubits it store 16 numbers". In the same way if we imply 5 qubits it store 32 numbers. Thus we have a simple sequence of  $2^n$ , with  $n$  as the number of cubits. Subsequently, seemingly, a quantum computer is able to do "massive amounts of parallel processing".

However it is not as simple as it , since as we have seen in the case of digital computers the electronic switches and Boolean logic worked together to solve storage and performance problems. Thus here the question is how we can store information and operate instructions.

One of the problems is the so called de-coherence which is the same as collapsing the superposed states or qubits to classical states or bits due to interaction with the environment. In fact a quantum computer must "operate with the de-coherence time, typically nanoseconds".

As we mentioned before, another problem is in the way of accessing to required information. "Measuring the state of a *quantum register* would simply collapse its wave function and give one of the eight possible numbers as in a classical register".

Therefore, in quantum computing always of the time we have to deal with destructive and instructive states interference based on probabilistic laws. Consequently, it is natural to have numerous **quantum algorithms** for registering required information based on these quantum effects.

One of the most famous effort to have an efficient quantum algorithm was in 1994, Peter Shor and known as **Shor's algorithm** in factorization of large numbers. Then the in 1996, Grover proposed an  $\sqrt{n}$  order algorithm for searching the data in unstructured database. In 1998, an Austrian scientists, Omer, proposed a quantum language called QCL based on C digital programming language for quantum algorithm simulation. Also in this year, the first quantum computer were gave birth by, Gershenfeld and Chuang at MIT. It was a 2-qubits quantum computer working based on a nuclear magnetic resonance. After this success, IBM introduced a 7-qubit computer system in

2001 using Shor's Algorithm and nuclear magnetic resonance (quantum) effect. Today people are working on 10-qubits systems and more.

Until today quantum computers have changed a lot and they are approaching to their real appearance and practice. For example, people employed by companies such as IBM or Google are working on the physical systems of quantum computers. These systems may be applied to "store qubits and to work with entangled registers". They "include ions held in an ion trap, superconducting quantum devices, quantum dots in semiconductors, and nuclear magnetic resonance".

Now we go on with the **programming**. They execute by computers logical hardware after translation from programming language into executable commands. A **procedural** or **imperative program** is set of instructions for computer to do a procedure and end in result. While "a **nonprocedural program** specifies constraints that must be satisfied by the results that are produced but does not specify the procedure by which these results should be obtained; such a procedure must be determined by a problem-solving **shell** based on the defined constraints".

We mentioned before to one of the earlier digital computers called ENIAC. Now we say; von Neumann made several alterations to this system in 1948 because of its simultaneous operation and performance which make the programming process very difficult. He did this using fixed connection cables for switches of control code and a converter code to enable serial operation. He also introduced architecture of modern computers using stored programs. In fact, the computers we use are Neumann-type computers. They have two characteristics. Firstly, "programs and data are stored in a computer". Secondly, "computation is performed sequentially by the instructions of a program".

Thus a program "is a set of machine instructions indicating the procedure of a computation. Data are objects, e.g., numbers, characters and strings, needed for computation".

However Allen Turing proposed one of the most important theories of computation called the Turing Machine in 1937. He also proposed a computer with stored program called, *Automatic Computing Engine* in 1946.

There is also a test called **Turing test** for thinking ability of thinking or computing machine.

As we mentioned earlier, computation plans such Ada, for early thinking machines, are the starting

point for the birth of programming languages. The task that modern computers today are also operating on them. Those early starting languages comes to assembly languages, and high-level procedural languages such as FORTRAN and COBOL. After them we have, some later high-level languages such as Pascal and C. then we come to some object-oriented languages such as C++ and Java. However all programming languages, again like their earlier models, are languages for doing a function i.e. Functional programming languages and languages for fundamental design of computing system i.e. logical programming languages.

FORTAN programming language, was introduced by IBM in 1957 as the first scientific programming language which is still popular. FORTRAN 77, BASIC, and COBOL were among the first programming languages applied to computing machines in 1970s and 80s.

In 1973 Gary Kildall developed the first high-level programming language for a microprocessor called Intel 4004. This high-level language i.e. PL/M equipped the programmers to write applications for microprocessors. PL/M also brought floppy disc drive and personal computers to the scene. One of the first personal computers

was Xerox Alto in 1973, as showed in the figure below.



With this systems the first Ethernets also introduced in 1975.

**First level** languages were streams of binary numbers 0 and 1 which were directly executed on the computer. They, do not need an assembler to "convert from a high-level language or assembly language into the machine code". **Second levels** are low-level assembly languages. They need an assembler to convert the assembly code into the machine codes for running on particular computing machine. The **third levels**, are high-level general-purpose programming languages such as ALGOL, COBOL, Pascal, FORTRAN, C,

C++, and Java. They are applied to business, science and other general uses. "They are designed to be easier for a human to understand and include features such as named variables, conditional statements, iterative statements, assignment statements and data structures".

In fact third level languages were focused on "how something is done" i.e. they were *imperative language* types. However later they looked at "what needs to be done" i.e. they became **object-oriented**. Thus they considered on particular problem to solve instead of assembly issues related to low level languages. Java and C++ are among such languages.

**Fourth levels**, are similar to third level programming languages but automated. To program automatically they use, report and form generators. The earlier "take a description of the data format and the report that is to be created and then automatically generate a program to produce the report". The latter, "are used to generate programs to manage online interactions with the application system users".

The **fifth level** languages are design to make the computer system solve the problem automatically. These languages use in research purposes such as artificial intelligence. They use some constraints

on the programming language instead of applying computational algorithm. Thus one can say they are logical programming languages. OPS5, Mercury and Prolog are examples of this level.

We mentioned Ethernet above which is a local network of computers. Now we consider the revolution of Internet. The designer of mechanical differential analyzer, Vannevar Bush also had an idea of World Wide Web in the 1940s in his article *As We May Think*. Licklider, 1960s proposed that "everyone around the globe would be interconnected and able to access programs and data at any site from anywhere". Then the first experience of a wide area computer network called, ARPANET was introduced in 1966, using the first common standards in data representation called ASCII (i.e. American Standard Code for Information Interchange) and telephone lines between MIT and Santa Monica. ARPA is an abbreviation for, Advanced Research Projects Agency belongs to, defense department of United States. ARPANET was in a huge success in 1972. After this extension, there was a need for a *network-to-network connection protocol*. Thus an international network-working group (INWG) was designed in 1973 to connect the networks to each other. For this purpose, a new protocol standards called *Transport Control Protocol*

(TCP) and the *Internet Protocol* (IP) were introduced. "TCP details how information is broken into packets and reassembled on delivery, whereas IP is focused on sending the packet across the network". Therefore users could send and transfer electronic mail and files on the basis of a common TCP/IP protocol. In fact there existed over 2000 hosts on the TCP/IP enabled Internet network by the mid-1980s with nodes in Canada and several European countries.

Along with these interesting inventions then again physics came to the scene by CERN which is a European center for nuclear research located in Switzerland. A visitor called Tim Berners-Lee spend a period of time there. He invented the World Wide Web, while working on an efficient way to share information among scientists. He did this in 1990 combining Internet; hypertext and the mouse using Universal Resource Locator URL. It is a standard address system for every web page which each of them are accessible using a protocol, called, Hypertext Transfer Protocol HTTP. The format of these pages is the well-known format called, hypertext markup language HTML. These pages are visible for user by a web browser.

Today web browser such as google and Yahoo along with the social networks such as Facebook and Twitter are working on these efforts of making an international network of computer systems. However all of this efforts from abacus to ENIAC, PC and a smartphone are working on numerical calculations and this is in continuation in today. Thus more advanced numerical calculation and more efficient matter building a computer gives us more incredible tools for communication. In fact, computing abilities of modern computing and communication devices are based on their power to "very large sets of data, using ever increasing capacities of disk storage and database software".

Thus when we are using, say, an app, we may think that now it is working based on "a complex network of servers, routers, databases, fiber-optic cables, satellites, mass storage arrays, and sophisticated software".

Now we go on and finish with some examples of programming languages. We consider FORTRAN-90 and Python. FORTRAN-90 in calculating the surface area of a cylinder as;

**program** cylinder

*! Calculate the surface area of a cylinder.*

*!*

*! Declare variables and constants.*

*! constants=pi*

*! variables=radius squared and height*

**implicit none**    *! Require all variables to be explicitly declared*

integer :: ierr

character(1) :: yn

real :: radius, height, area

real, **parameter** :: pi = 3.141592653589793

interactive\_loop: **do**

*! Prompt the user for radius and height*

*! and read them.*

**write** (\*,\*) 'Enter radius and height.'

**read** (\*,\*,iostat=ierr) radius,height

*! If radius and height could not be read from input,*

*! then cycle through the loop.*

**if** (ierr /= 0) **then**

**write**(\*,\*) 'Error, invalid input.'

**cycle** interactive\_loop

**end if**

*! Compute area. The \*\* means "raise to a power."*

`area = 2*pi * (radius**2 + radius*height)`

*! Write the input variables (radius, height)*

*! and output (area) to the screen.*

`write (*, '(1x,a7,f6.2,5x,a7,f6.2,5x,a5,f6.2)') &  
'radius=',radius,'height=',height,'area=',area`

`yn = ''`

`yn_loop: do`

`write(*,*) 'Perform another calculation? y[n]'`

`read(*, '(a1)') yn`

`if (yn=='y' .or. yn=='Y') exit yn_loop`

`if (yn=='n' .or. yn=='N' .or. yn==' ') exit`

`interactive_loop`

`end do yn_loop`

`end do interactive_loop`

`end program cylinder`

Now a sample of python in some more examples below;

# Program for computing the height of a ball in vertical motion

v0 = 5 # Initial velocity

```

g = 9.81 # Acceleration of gravity
t = 0.6 # Time
y = v0*t - 0.5*g*t**2 # Vertical position
print y

```

Implementations of the midpoint rule for a double integral and its corresponding functions are shown below;

```

def midpoint_triple1(g, a, b, c, d, e, f, nx, ny, nz):
    hx = (b - a)/float(nx)
    hy = (d - c)/float(ny)
    hz = (f - e)/float(nz)
    l = 0
    for i in range(nx):
        for j in range(ny):
            for k in range(nz):
                xi = a + hx/2 + i*hx
                yj = c + hy/2 + j*hy
                zk = e + hz/2 + k*hz
                l += hx*hy*hz*g(xi, yj, zk)
    return l

def midpoint(f, a, b, n):
    h = float(b-a)/n
    result = 0
    for i in range(n):
        result += f((a + h/2.0) + i*h)
    result *= h
    return result

def midpoint_triple2(g, a, b, c, d, e, f, nx, ny, nz):
    def p(x, y):
        return midpoint(lambda z: g(x, y, z), e, f, nz)

```

```

def q(x):
    return midpoint(lambda y: p(x, y), c, d, ny)
    return midpoint(q, a, b, nx)
def test_midpoint_triple():
    """Test that a linear function is integrated exactly."""
    def g(x, y, z):
        return 2*x + y - 4*z
    a = 0; b = 2; c = 2; d = 3; e = -1; f = 2
    import sympy
    x, y, z = sympy.symbols('x y z')
    I_expected = sympy.integrate(
        g(x, y, z), (x, a, b), (y, c, d), (z, e, f))
    for nx, ny, nz in (3, 5, 2), (4, 4, 4), (5, 3, 6):
        I_computed1 = midpoint_triple1(
            g, a, b, c, d, e, f, nx, ny, nz)
        I_computed2 = midpoint_triple2(
            g, a, b, c, d, e, f, nx, ny, nz)
    tol = 1E-14
    print I_expected, I_computed1, I_computed2
    assert abs(I_computed1 - I_expected) < tol
    assert abs(I_computed2 - I_expected) < tol
    if __name__ == '__main__':
        test_midpoint_triple().

```

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